

CLASSIFYING SPACES AND MODULI SPACES OF ALGEBRAS OVER A PROP

SINAN YALIN

ABSTRACT. The purpose of this article is two-fold. First we show that a weak equivalence between cofibrant props induces a weak equivalence between the associated classifying spaces of bialgebras. This statement generalizes to the prop setting a homotopy invariance result which is well known in the case of algebras over operads. The absence of model category structure on algebras over a prop leads us to introduce new methods to overcome this difficulty.

We also explain how our result can be extended to algebras over colored props in any symmetric monoidal model category tensored over chain complexes.

Then we provide a generalization of a theorem of Charles Rezk in the setting of algebras over a (colored) prop: we introduce the notion of moduli space of algebra structures over a prop, and prove that under certain conditions such a moduli space is the homotopy fiber of a map between classifying spaces.

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INTRODUCTION

The notion of a prop has been introduced by MacLane ([16]) in algebra. The name prop is actually an acronym for “product and permutation”. Briefly, a prop P is a double sequence of objects $P(m, n)$ whose elements represent operations with m inputs and n outputs.

Certain categories of algebras, like associative, Poisson or Lie algebras, have a structure which is fully determined by operations with a single output. These categories are associated to props P of a certain form, where operations in components $P(m, 1)$ generate the prop. Boardman and Vogt coined the name categories of operators of standard form to refer to props of this particular form [1]. Peter May introduced the axioms of operads to deal with the components $P(m, 1)$ which define the core of such prop structures [19]. The work of these authors was initially motivated by the theory of iterated loop spaces, in topology (see [2] and [19]). Operads have now proved to be a powerful device to handle a variety of algebraic structures occurring in many branches of mathematics.

However, if one wants to deal with bialgebras it becomes necessary to use general props instead of operads. Important examples appeared especially in mathematical physics and string topology : the Frobenius bialgebras (whose category is equivalent to the category of two-dimensional topological quantum field theories), the topological conformal field theories (which are algebras over the chain Segal prop), or the Lie bialgebras introduced by Drinfeld in quantization theory are categories of bialgebras associated to props.

The purpose of this article is to set up a theory for the homotopical study of bialgebras over a (possibly colored) prop. In a seminal series of papers at the beginning of the 80's, Dwyer and Kan investigated the simplicial localization of categories. They proved that the simplicial localization gives a good device to capture secondary homology structures usually defined in the framework of Quillen's model categories ([5]). An important homotopy invariant of a model category is its classifying space, defined as the nerve of its subcategory of weak equivalences. Dwyer and Kan studied homotopy invariance properties of such classifying spaces.

The algebras over an operad in a model category form themselves, under suitable assumptions, a model category. A consequence of usual results about model categories is that the classifying space of such a category is homotopy invariant up to the weak homotopy type of the underlying operad. Unfortunately, there is no model category structure on the algebras over a prop in general. We can not handle our motivating examples of bialgebras occurring in mathematical physics and string topology by using this approach, and we aim to overcome this difficulty.

The basic problem is to compare categories of bialgebras over a prop. In order to bypass difficulties due to the absence of model structure on bialgebras, our overall strategy is to stay at the prop level as far as possible, and to use factorization and lifting properties in the model category of props. The structure of an algebra over a prop P can be encoded by a prop morphism $P \rightarrow \text{End}_A$, where End_A is the endomorphism prop associated to A . One can construct a version of endomorphisms props modeling P -algebras structures on diagrams. We can notably use these diagrams endomorphisms props to define path objects in the category of P -algebras. But we need an analogue of this device for a variable P -algebra A , not a fixed object. The idea is to perform such a construction on the abstract prop P itself before moving to endomorphisms props. Combining this P -modified prop feature with lifting and factorization techniques, we endow the category of P -algebras with functorial path objects.

Consequently, the first main outcome of our study is the following homotopy invariance theorem. Let $Ch_{\mathbb{K}}$ be the category of \mathbb{Z} -graded chain complexes over a field \mathbb{K} of characteristic zero. Let $(Ch_{\mathbb{K}})^P$ be the category of bialgebras associated to a prop P in this category, and $w(Ch_{\mathbb{K}})^P$ its subcategory obtained by restricting to morphisms which are weak equivalences in $Ch_{\mathbb{K}}$. Our result reads:

Theorem 0.1. *Let $\varphi : P \xrightarrow{\sim} Q$ be a weak equivalence between two cofibrant props. The map φ gives rise to a functor $\varphi^* : w(Ch_{\mathbb{K}})^Q \rightarrow w(Ch_{\mathbb{K}})^P$ which induces a weak equivalence of simplicial sets $\mathcal{N}\varphi^* : \mathcal{N}w(Ch_{\mathbb{K}})^Q \xrightarrow{\sim} \mathcal{N}w(Ch_{\mathbb{K}})^P$.*

We can withdraw the hypothesis about the characteristic of \mathbb{K} if we suppose that P is a prop with non-empty inputs or outputs (see definition 1.12 and theorem 1.13). We explain in 2.7 how to extend theorem 0.1 to the case of a category tensored over $Ch_{\mathbb{K}}$. In section 4, we also briefly show that the proof of theorem 0.1 extends readily to the colored props context if we suppose \mathbb{K} to be of characteristic zero (this hypothesis is needed to put a model category structure on colored props in Ch_K , see the work of Johnson and Yau[14]).

Rezk considers in his thesis [22] the moduli spaces $\mathcal{A}\{X\}$ of algebras structures over an operad \mathcal{A} , which are simplicial sets whose 0-simplexes are operad morphisms $\mathcal{A} \rightarrow \text{End}_X$ representing all the \mathcal{A} -algebra structures on a given object X . The first main result of his thesis, proved in the case of simplicial sets and simplicial

modules, is that under certain conditions such a moduli space is the homotopy fiber of a map between classifying spaces. Combining our theorem 0.1 with the technical proposition 3.1 proved in 3.2, we obtain a broad generalization of this result:

Theorem 0.2. (*Generalization of [22], theorem 1.1.5, in the case of props*). *Let P be a cofibrant prop defined in $Ch_{\mathbb{K}}$ and X a fibrant and cofibrant object of $Ch_{\mathbb{K}}$. Then the commutative square*

$$\begin{array}{ccc} P\{X\} & \longrightarrow & \mathcal{N}(w(Ch_{\mathbb{K}})^P) \\ \downarrow & & \downarrow \\ \{X\} & \longrightarrow & \mathcal{N}(wCh_{\mathbb{K}}) \end{array}$$

is a homotopy pullback of simplicial sets.

As previously, we can extend this result to colored props and categories tensored over $Ch_{\mathbb{K}}$.

Remark 0.3. We do not address the case of simplicial sets. However, theorem 1.4 in [14] endows the algebras over a colored prop in simplicial sets with a model category structure. Moreover, the free algebra functor exists in this case. Therefore one can transpose the methods used in the operadic setting to obtain a simplicial version of theorem 0.1. Theorem 0.2 in simplicial sets can be proved by following step by step Rezk's original proof. We also conjecture that our results have a version in simplicial modules which follows from arguments similar to ours.

We would like to emphasize some links with two other objects encoding algebraic structures and their deformations. Theorem 0.2 asserts that we can use a function space of props, the moduli space $P\{X\}$, to determine classifying spaces of categories of algebras over props $\mathcal{N}(w(Ch_{\mathbb{K}})^P)$. The homotopy groups of this moduli space, in turn, can be approached by means of a Bousfield-Kan type spectral sequence. The E_2 page of this spectral sequence is identified with the cohomology of certain deformation complexes. These complexes have been studied by Markl in [18] and Merkulov and Vallette in [20]. Both papers prove the existence of an L_{∞} structure on such complexes which generalizes the *intrinsic Lie bracket* of Schlessinger and Stasheff [23]. We aim to apply this spectral sequence technique and provide new results about the deformation theory of bialgebras in an ongoing work. To complete this outlook, let us point out that Ciocan-Fontanine and Kapranov used a similar approach to that of Rezk in [3] to define a derived moduli space of algebras structures in the formalism of dg schemes. This construction remains valid for props and formalizes, in the context of algebraic geometry, the idea that the deformation complexes form the tangent space of the moduli space $P\{X\}$.

Organization: the overall setting is reviewed in section 1. We recall some definitions about symmetric monoidal categories over a base category and axioms of monoidal model categories. Then we introduce the precise definition of props and algebras over a prop. We conclude these recollections with a fundamental result, the existence of a model structure on the category of props.

The heart of this paper consists of sections 2 and 3, devoted to the proofs of theorem 0.1 and proposition 3.1. The proof of theorem 0.1 is quite long and has been

consequently divided in several steps. Subsection 2.1 gives a sketch of our main arguments. In 2.3, 2.4 and 2.5 we define particular props called P -modified endomorphism props, which allow us to build a functorial path object in P -algebras. In 2.6 we give a proof of theorem 0.1. To conclude section 2 we generalize theorem 0.1 to categories tensored over $Ch_{\mathbb{K}}$. In 3.1, we prove that under suitable assumptions a diagram category inherits a monoidal model category structure from the base category. The transfer of model structure is a well known result, but the compatibility with the symmetric monoidal structure over the base category does not seem to appear in the litterature. This general result allows us to give a proof of proposition 3.1 in 3.2. Then we quickly explain in section 4 the extension of our arguments to colored props.

Finally, we show in section 5 how theorem 0.1 and proposition 3.1 fit in an adaptation of Rezk's proof of theorem 1.1.5 in [22]. We thus obtain theorem 0.2. This adaptation need some preliminary results given in 5.1 and 5.2. The last remark of 5.3 shows how to recover theorem A of Fresse in [7] as a consequence of theorem 0.2.

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1. RECOLLECTIONS AND GENERAL RESULTS

1.1. Symmetric monoidal categories over a base category.

Definition 1.1. Let \mathcal{C} be a symmetric monoidal category. A *symmetric monoidal category over \mathcal{C}* is a symmetric monoidal category $(\mathcal{E}, \otimes, 1_{\mathcal{E}})$ endowed with an external tensor product $\otimes : \mathcal{C} \times \mathcal{E} \rightarrow \mathcal{E}$ satisfying the following natural unit, associativity and symmetry constraints:

- (1) $\forall X \in \mathcal{E}, 1_{\mathcal{C}} \otimes X \cong X$,
- (2) $\forall X \in \mathcal{E}, \forall C, D \in \mathcal{C}, (C \otimes D) \otimes X \cong C \otimes (D \otimes X)$,
- (3) $\forall C \in \mathcal{C}, \forall X, Y \in \mathcal{E}, C \otimes (X \otimes Y) \cong (C \otimes X) \otimes Y \cong X \otimes (C \otimes Y)$.

We will tacitly assume throughout the paper that all small limits and small colimits exist in \mathcal{C} and that the internal tensor product of \mathcal{C} preserves colimits in each variable. We assume the same hypotheses for \mathcal{E} , and suppose moreover that the external tensor product of \mathcal{E} also preserves colimits in each variable. This last condition implies the existence of an external hom bifunctor $Hom_{\mathcal{E}}(-, -) : \mathcal{E}^{op} \times \mathcal{E} \rightarrow \mathcal{C}$ satisfying an adjunction relation

$$\forall C \in \mathcal{C}, \forall X, Y \in \mathcal{E}, Mor_{\mathcal{E}}(C \otimes X, Y) \cong Mor_{\mathcal{C}}(C, Hom_{\mathcal{E}}(X, Y))$$

(so \mathcal{E} is naturally an enriched category over \mathcal{C}).

Examples.

(1) The differential graded \mathbb{K} -modules (where \mathbb{K} is a commutative ring) form a symmetric monoidal category over the \mathbb{K} -modules. This is the main category used in this paper.

(2) Any symmetric monoidal category \mathcal{C} forms a symmetric monoidal category over Set (the category of sets) with an external tensor product defined by

$$\begin{aligned} \otimes : Set \times \mathcal{C} &\rightarrow \mathcal{C} \\ (S, C) &\mapsto \bigoplus_{s \in S} C. \end{aligned}$$

(3) Let I be a small category ; the I -diagrams in a symmetric monoidal category \mathcal{C} form a symmetric monoidal category over \mathcal{C} . The external hom $Hom_{\mathcal{C}^I}(-, -) : \mathcal{C}^I \times \mathcal{C}^I \rightarrow \mathcal{C}$ is given by

$$Hom_{\mathcal{C}^I}(X, Y) = \int_{i \in I} Hom_{\mathcal{C}}(X(i), Y(i)).$$

This example will be useful especially in section 3.

Remark 1.2. If \mathcal{E} is a symmetric monoidal category over \mathcal{D} and \mathcal{D} a symmetric monoidal category over \mathcal{C} , then \mathcal{E} is a symmetric monoidal category over \mathcal{C} .

Proposition 1.3. *Let $F : \mathcal{D} \rightleftarrows \mathcal{E} : G$ be an adjunction between two symmetric monoidal categories over \mathcal{C} . If F preserves the external tensor product then F and G satisfy an enriched adjunction relation*

$$\mathrm{Hom}_{\mathcal{E}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, G(Y))$$

at the level of the external hom bifunctors (see proposition 1.1.16 in Fresse's lecture notes [8] for the proof).

We now deal with symmetric monoidal categories equipped with a model structure. We assume that the reader is familiar with the basics of model categories. We refer to the well written paper of Dwyer and Spalinski [4] for a complete and accessible introduction, and to Hirschhorn [11] and Hovey [13] for a comprehensive treatment. We just recall the axioms of symmetric monoidal model categories formalizing the interplay between the tensor and the model structure.

Definition 1.4. Let \mathcal{C} be a category with colimits and $F : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ a bifunctor. The *pushout-product* of two morphisms $f : A \rightarrow B \in \mathcal{A}$ and $g : C \rightarrow D \in \mathcal{B}$ is the morphism

$$(f_*, g_*) : F(A, D) \oplus_{F(A, C)} F(B, C) \rightarrow F(B, D)$$

given by the commutative diagram:

$$\begin{array}{ccc} F(A, C) & \xrightarrow{F(f, C)} & F(B, C) \\ \downarrow F(A, g) & & \downarrow \\ F(A, D) & \xrightarrow{\quad} & F(A, D) \oplus_{F(A, C)} F(B, C) \\ & \searrow F(f, D) & \nearrow (f_*, g_*) \\ & & F(B, D) \end{array} \quad .$$

Definition 1.5. (1) A *symmetric monoidal model category* is a symmetric monoidal category \mathcal{C} equipped with a model category structure such that the following axioms hold:

MM0. The unit object is cofibrant in \mathcal{C} .

MM1. The pushout-product $(i_*, j_*) : A \otimes D \oplus_{A \otimes C} B \otimes C \rightarrow B \otimes D$ of cofibrations $i : A \rightarrow B$ and $j : C \rightarrow D$ is a cofibration which is also acyclic as soon as i or j is so.

(2) Suppose that \mathcal{C} is a symmetric monoidal model category. A symmetric monoidal category \mathcal{E} over \mathcal{C} is a symmetric monoidal model category over \mathcal{C} if the axiom MM0 holds and the axiom MM1 holds for both the internal and external tensor products of \mathcal{E} .

Example: the category $Ch_{\mathbb{K}}$ of chain complexes over a field \mathbb{K} is our main working example of symmetric monoidal model category.

Lemma 1.6. *In a symmetric monoidal model category \mathcal{E} over \mathcal{C} the axiom MM1 for the external tensor product is equivalent to the following one:*

MM1'. *The morphism*

$$(i^*, p_*) : \mathrm{Hom}_{\mathcal{E}}(B, X) \rightarrow \mathrm{Hom}_{\mathcal{E}}(A, X) \times_{\mathrm{Hom}_{\mathcal{E}}(A, Y)} \mathrm{Hom}_{\mathcal{E}}(B, Y)$$

induced by a cofibration $i : A \rightarrowtail B$ and a fibration $p : X \twoheadrightarrow Y$ is a fibration in \mathcal{C} which is also acyclic as soon as i or p is so (cf. lemma 4.2.2 in [13]).

In good cases, the fact that the internal tensor product of \mathcal{E} preserves colimits in each variable implies the existence of an internal hom bifunctor. The axiom MM1 for the internal tensor product is in the same way equivalent to a “dual” axiom MM1’.

To conclude this section, we define additional axioms (introduced in [7]) that we will need to prove proposition 3.1.

Definition 1.7. We say that a symmetric monoidal category \mathcal{C} satisfies the *limit monoid axioms* when the following properties hold:

LM1 (final monoid axiom). Let $*$ be the terminal object of \mathcal{C} , the natural morphism $* \otimes * \rightarrow *$ is an isomorphism.

LM2 (cartesian monoid axiom). For every fibration of the form

$$(f, g) : S \twoheadrightarrow X \times_T Y$$

and every cofibrant-fibrant object Z , the morphism

$$(f \otimes Z, g \otimes Z) : S \otimes Z \twoheadrightarrow X \otimes Z \times_{T \otimes T} Y \otimes Z$$

is also a fibration.

Example: the category $Ch_{\mathbb{K}}$ of chain complexes over a field \mathbb{K} is an example of category satisfying these axioms. It will be again our main example of such a category in our paper.

Proposition 1.8. (cf. [7], proposition 6.7) *The following properties hold in any symmetric monoidal category satisfying the limit monoid axioms:*

- (1) *If X is cofibrant-fibrant, then $X^{\otimes n}$ is fibrant for every $n \in \mathbb{N}$.*
- (2) *If $p : X \twoheadrightarrow Y$ is a fibration and Y is cofibrant-fibrant, then $p^{\otimes n} : X^{\otimes n} \twoheadrightarrow Y^{\otimes n}$ is also a fibration.*
- (3) *If $p : Z \twoheadrightarrow X \times Y$ is a fibration and X, Y are cofibrant-fibrant objects, then*

$$p^{\otimes n} : Z^{\otimes n} \twoheadrightarrow X^{\otimes n} \times Y^{\otimes n}$$

is also a fibration.

1.2. On Σ -bimodules, props and algebras over a prop. Let \mathcal{C} be a symmetric monoidal category admitting all small limits and small colimits, whose tensor product preserves colimits and endowed with an internal hom bifunctor. Let \mathbb{B} be the category having the pairs $(m, n) \in \mathbb{N}^2$ as objects together with morphisms sets such that:

$$Mor_{\mathbb{B}}((m, n), (p, q)) = \begin{cases} \Sigma_m^{op} \times \Sigma_n, & \text{if } (p, q) = (m, n), \\ \emptyset & \text{otherwise.} \end{cases}$$

The Σ -biobjects in \mathcal{C} are the \mathbb{B} -diagrams in \mathcal{C} . So a Σ -biobject is a double sequence $\{M(m, n) \in \mathcal{C}\}_{(m, n) \in \mathbb{N}^2}$ where each $M(m, n)$ is equipped with a right action of Σ_m and a left action of Σ_n commuting with each other. Let \mathbb{A} be the discrete category of pairs $(m, n) \in \mathbb{N}^2$. We have an obvious forgetful functor $\phi^* : \mathcal{C}^{\mathbb{B}} \rightarrow \mathcal{C}^{\mathbb{A}}$. This functor has a left adjoint $\phi_! : \mathcal{C}^{\mathbb{A}} \rightarrow \mathcal{C}^{\mathbb{B}}$ defined on objects by

$$\begin{aligned} \forall M \in \mathcal{C}^{\mathbb{A}}, \forall (m, n) \in \mathbb{N}^2, \phi_! M(m, n) &= 1_{\mathcal{C}}[\Sigma_n \times \Sigma_m^{op}] \otimes M(m, n) \\ &\cong \bigoplus_{\Sigma_n \times \Sigma_m^{op}} M(m, n). \end{aligned}$$

Definition 1.9. (1) Let \mathcal{C} be a symmetric monoidal category. A *prop* in \mathcal{C} is a symmetric monoidal category P , enriched over \mathcal{C} , with \mathbb{N} as object set and the tensor product given by $m \otimes n = m + n$ on objects. Let us unwrap this definition. Firstly we see that a prop is a Σ -biobject. Indeed, the group Σ_m acts on $m = 1 + \dots + 1 = 1^{\otimes m}$ and the group Σ_n^{op} acts on $n = 1 + \dots + 1 = 1^{\otimes n}$ by permuting the variables at the morphisms level. A prop is endowed with horizontal products

$$\circ_h : P(m_1, n_1) \otimes P(m_2, n_2) \rightarrow P(m_1 + m_2, n_1 + n_2)$$

which are defined by the tensor product of homomorphisms, since $P(m_1 \otimes m_2, n_1 \otimes n_2) = P(m_1 + m_2, n_1 + n_2)$ by definition of the tensor product on objects. It also admits vertical composition products

$$\circ_v : P(k, n) \otimes P(m, k) \rightarrow P(m, n)$$

corresponding to the composition of homomorphisms, and units $1 \rightarrow P(n, n)$ corresponding to identity morphisms of the objects $n \in \mathbb{N}$ in P . These operations satisfy relations coming from the axioms of symmetric monoidal categories. We refer the reader to Enriquez and Etingof [6] for an explicit description of props in the context of modules over a ring. We denote by \mathcal{P} the category of props.

Another construction of props is given in [14]: props are defined there as \boxtimes_h -monoids in the \boxtimes_v -monoids of colored Σ -biobjects, where \boxtimes_h and \boxtimes_v denote respectively a horizontal composition product and a vertical composition product.

Appendix A of [7] provides a construction of the free prop on a Σ -biobject. The free prop functor is left adjoint to the forgetful functor:

$$F : \mathcal{C}^{\mathbb{B}} \rightleftarrows \mathcal{P} : U.$$

Definition 1.10. (1) To any object X of \mathcal{C} we can associate an *endomorphism prop* End_X defined by

$$End_X(m, n) = Hom_{\mathcal{C}}(X^{\otimes m}, X^{\otimes n}).$$

The actions of the symmetric groups are the permutations of the input variables and of the output variables, the horizontal product is the tensor product of homomorphisms and the vertical composition product is the composition of homomorphisms. The units $1_{\mathcal{C}} \rightarrow Hom_{\mathcal{C}}(X^{\otimes n}, X^{\otimes n})$ represent $id_{X^{\otimes n}}$.

(2) An *algebra over a prop* P , or P -algebra, is an object $X \in \mathcal{C}$ equipped with a prop morphism $P \rightarrow End_X$.

We can also define a P -algebra in a symmetric monoidal category over \mathcal{C} :

Definition 1.11. Let \mathcal{E} be a symmetric monoidal category over \mathcal{C} .

(1) The endomorphism prop of $X \in \mathcal{E}$ is given by $End_X(m, n) = Hom_{\mathcal{E}}(X^{\otimes m}, X^{\otimes n})$ where $Hom_{\mathcal{E}}(-, -)$ is the external hom bifunctor of \mathcal{E} .

(2) Let P be a prop in \mathcal{C} . A P -algebra in \mathcal{E} is an object $X \in \mathcal{E}$ equipped with a prop morphism $P \rightarrow End_X$.

We unwrap the definition in the case of a diagram category over \mathcal{E} : let $\{X_i\}_{i \in I}$ be a I -diagram in \mathcal{E} , then

$$End_{\{X_i\}_{i \in I}} = \int_{i \in I} Hom_{\mathcal{E}}(X_i^{\otimes m}, X_i^{\otimes n}).$$

This end can equivalently be defined as a coreflexive equalizer

$$End_{\{X_i\}}(m, n) \longrightarrow \prod_{i \in I} Hom_{\mathcal{E}}(X_i^{\otimes m}, X_i^{\otimes n}) \xrightleftharpoons[d_1]{d_0} \prod_{u: i \rightarrow j \in mor(I)} Hom_{\mathcal{E}}(X_i^{\otimes m}, X_j^{\otimes n})$$

s_0

where d_0 is the product of the maps

$$u_* : Hom_{\mathcal{E}}(X_i^{\otimes m}, X_i^{\otimes n}) \rightarrow Hom_{\mathcal{E}}(X_i^{\otimes m}, X_j^{\otimes n})$$

induced by the morphisms $u : i \rightarrow j$ of I and d_1 is the product of the maps

$$u^* : Hom_{\mathcal{E}}(X_j^{\otimes m}, X_j^{\otimes n}) \rightarrow Hom_{\mathcal{E}}(X_i^{\otimes m}, X_j^{\otimes n})$$

The section s_0 is the projection on the factors associated to the identities $id : i \rightarrow i$.

This construction is functorial in I : given a J -diagram $\{X_j\}_{j \in J}$, every functor $\alpha : I \rightarrow J$ gives rise to a prop morphism $\alpha^* : End_{\{X_j\}_{j \in J}} \rightarrow End_{\{X_{\alpha(i)}\}_{i \in I}}$.

1.3. The semi-model category of props. Suppose that \mathcal{C} is a cofibrantly generated symmetric monoidal model category. The category of Σ -biobjects $\mathcal{C}^{\mathbb{B}}$ is a diagram category over \mathcal{C} , so it inherits a cofibrantly generated model category structure. The weak equivalences and fibrations are defined componentwise. The generating (acyclic) cofibrations are given by $i \otimes \phi_! G_{(m,n)}$, where $(m, n) \in \mathbb{N}^2$ and i ranges over the generating (acyclic) cofibrations of \mathcal{C} . Here \otimes is the external tensor product of $\mathcal{C}^{\mathbb{B}}$ and $G_{(m,n)}$ is the double sequence defined by

$$G_{(m,n)}(p, q) = \begin{cases} 1_{\mathcal{C}}, & \text{if } (p, q) = (m, n), \\ 0 & \text{otherwise.} \end{cases}$$

We can also see this result as a transfer of cofibrantly generated model category structure via the adjunction $\phi_! : \mathcal{C}^{\mathbb{A}} \rightleftarrows \mathcal{C}^{\mathbb{B}} : \phi^*$ (this is a propic version of proposition 11.4.A in [8]). The question is to know if the adjunction $F : \mathcal{C}^{\mathbb{B}} \rightleftarrows \mathcal{P} : U$ transfer this model category structure to the props. In the general case it works only with the subcategory \mathcal{P}_0 of props with non-empty inputs or outputs and does not give a full model category structure. We give the precise statement in theorem 1.13.

Definition 1.12. A Σ -biobject M has *non-empty inputs* if it satisfies

$$M(0, n) = \begin{cases} 1_{\mathcal{C}}, & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We define in a symmetric way a Σ -biobject with *non-empty outputs*. The category of Σ -biobjects with non-empty inputs is noted $\mathcal{C}_0^{\mathbb{B}}$.

The composite adjunction

$$\mathcal{C}^{\mathbb{A}} \rightleftarrows \mathcal{C}^{\mathbb{B}} \rightleftarrows \mathcal{P}$$

restricts to an adjunction

$$\mathcal{C}_0^{\mathbb{A}} \rightleftarrows \mathcal{C}_0^{\mathbb{B}} \rightleftarrows \mathcal{P}_0.$$

We define the weak equivalences (respectively fibrations) in \mathcal{P}_0 componentwise, i.e their images by the forgetful functor $U : \mathcal{P}_0 \rightarrow \mathcal{C}_0^{\mathbb{A}}$ are weak equivalences (respectively fibrations) in $\mathcal{C}_0^{\mathbb{A}}$. We define the generating (acyclic) cofibrations as the images under the free prop functor of the generating (acyclic) cofibrations of $\mathcal{C}_0^{\mathbb{B}}$. We have the following result:

Theorem 1.13. *(cf. [7], theorem 4.9) Let \mathcal{C} be a cofibrantly generated symmetric monoidal model category. The category \mathcal{P}_0 of props with non-empty inputs (or outputs) equipped with the classes of weak equivalences, fibrations and cofibrations of 1.3 forms a semi-model category. Moreover the forgetful functor $U : \mathcal{P}_0 \rightarrow \mathcal{C}_0^{\mathbb{A}}$ preserves cofibrations with cofibrant domain.*

A semi-model category structure is a slightly weakened version of model category structure: the lifting axioms work only with cofibrations with cofibrant domain, and the factorization axioms work only on a map with cofibrant domain (see the relevant section of [7]). The notion of a semi-model category is sufficient to do homotopy theory. In certain categories we recover a full model structure on the whole category of props:

Theorem 1.14. *(cf. [7], theorem 5.5) If the base category \mathcal{C} is the category of dg-modules over a ring \mathbb{K} such that $\mathbb{Q} \subset \mathbb{K}$, simplicial modules over a ring, simplicial sets or topological spaces, then the definition of theorem 1.13 provides \mathcal{P} with a model category structure.*

2. PROOF OF THEOREM 0.1

The purpose of this section is to establish theorem 0.1. We give the details of our arguments in the case $\mathcal{E} = \mathcal{C} = Ch_{\mathbb{K}}$ (the \mathbb{Z} -graded chain complexes over a field \mathbb{K} of characteristic zero). We explain briefly afterwards the generalization of these arguments when \mathcal{E} is a cofibrantly generated symmetric monoidal model category over $Ch_{\mathbb{K}}$.

2.1. Statement of the result and outline of the proof. In the work of Dwyer-Kan [5], the classifying space of a model category \mathcal{M} is the simplicial set $\mathcal{N}(w\mathcal{M})$ where \mathcal{N} is the simplicial nerve functor and $w\mathcal{M}$ is the subcategory of weak equivalences of \mathcal{M} . In the case of \mathcal{E}^P (the P -algebras in \mathcal{E} for a prop P defined in \mathcal{C}) we use the expression of classifying space to refer to the simplicial set $\mathcal{N}w(\mathcal{E}^{cf})^P$, where $w(\mathcal{E}^{cf})^P$ is the subcategory of P -algebra morphisms whose underlying morphisms in \mathcal{E} are weak equivalences between fibrant-cofibrant objects. In the operadic context, algebras over operads satisfy the following useful property: a weak equivalence between two cofibrant operads induces a weak equivalence between their associated classifying spaces of algebras. The proof of this result works in two steps: firstly, one show the existence of an adjunction between the two categories of algebras induced by the operads morphism, then one prove that this adjunction forms actually a Quillen equivalence. Such a method fails in the prop setting: there is no free algebra functor, and accordingly a model structure does not exist on the category of algebras over a prop, except in some particular cases such as simplicial sets (see [14]). So the difficult part is to deal with this absence of model structure to get a

similar result for algebras over props. Therefore, our method is entirely different from this one. The crux of our proof is given by the following statement:

Theorem 2.1. *Let P be a cofibrant prop. The mappings $\mathcal{N}\varphi^*, \mathcal{N}\psi^* : \mathcal{N}w(\mathcal{E}^{cf})^P \rightrightarrows \mathcal{N}w(\mathcal{E}^{cf})^P$ associated to homotopic prop morphisms $\varphi, \psi : P \rightrightarrows P$ are homotopic in $sSet$.*

Let us outline the main steps of the proof of theorem 2.1 in the case $\mathcal{E} = \mathcal{C} = Ch_{\mathbb{K}}$. The idea is to construct a zigzag of natural transformations $\varphi^* \leftarrow Z \rightarrow \psi^*$, where Z is a functorial path object in $Ch_{\mathbb{K}}^P$. We proceed as follows. We use functional notations $\mathcal{Y}(X)$, $\mathcal{Z}(X)$ and $\mathcal{V}(X)$ to refer to diagrams functorially associated to an object X which, in our constructions, ranges within (some subcategory of) Ch_K . We first consider the functorial path object diagram associated to any X in $Ch_{\mathbb{K}}$

$$\mathcal{Y}(X) :$$

and its subdiagram $\mathcal{Z}(X) = \{X_0 \xleftarrow{\sim} Z(X) \xrightarrow{\sim} X_1\}$. We prove that the natural P -action existing on the diagram

$$\mathcal{V}(X) :$$

extends to a natural P -action on $\mathcal{Y}(X)$. For this aim, we consider “ P -modified endomorphism props”, which are built by replacing all the operations $X^{\otimes m} \rightarrow X^{\otimes n}$ in the endomorphism prop of a given diagram by operations of $P(m, n)$. We use notations $End_{\mathcal{Y}(P)}$, $End_{\mathcal{Z}(P)}$ and $End_{\mathcal{V}(P)}$ to refer to these P -modified endomorphism props. We verify that these constructions give rise to props acting naturally on the endomorphism prop of the associated diagram. We use these P -modified endomorphism props to give a P -action on the zigzag of endofunctors $Id \xleftarrow{\sim} Z \xrightarrow{\sim} Id$. We check that we retrieve the action given by φ and ψ on the extremity of this zigzag. We thus have a zigzag connecting φ^* and ψ^* and yielding the desired homotopy between $\mathcal{N}\varphi^*$ and $\mathcal{N}\psi^*$.

Our argument line is divided in two steps. For every $X \in Ch_{\mathbb{K}}^P$, we have $End_{\mathcal{V}(X)} \cong End_X$ so the morphism $P \rightarrow End_X$ trivially induces a morphism $P \rightarrow \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{V}(X)}$. We use ends to obtain a functorial version of our endomorphism props of diagrams (we refer to MacLane [15] for the definition of ends and coends). In our first step we build a diagram

$$\begin{array}{ccc}
 \text{End}_{\mathcal{Y}(P)} & \longrightarrow & \int_{X \in Ch_{\mathbb{K}}^P} \text{End}_{\mathcal{Y}(X)} \\
 \uparrow \pi & \sim & \downarrow \\
 P & \xrightarrow{=} & P \longrightarrow \int_{X \in Ch_{\mathbb{K}}^P} \text{End}_{\mathcal{Y}(X)}
 \end{array}$$

In $Ch_{\mathbb{K}}$, the endomorphism prop $\text{End}_{\mathcal{Y}(X)}$ is built via the two following pullbacks:

$$\begin{array}{ccc}
 \text{End}_{\mathcal{Y}(X)} & \longrightarrow & \text{End}_{\mathcal{Z}(X)} \\
 \downarrow & & \downarrow s^* \circ pr \\
 \text{End}_X & \xrightarrow{s_*} & \text{Hom}_{X, \mathcal{Z}(X)}
 \end{array}$$

and

$$\begin{array}{ccc}
 \text{End}_{\mathcal{Z}(X)} & \longrightarrow & \text{End}_{X_0} \times \text{End}_{X_1} \\
 \downarrow & & \downarrow d_0^* \times d_1^* \\
 \text{End}_{\mathcal{Z}(X)} & \xrightarrow{(d_0, d_1)_*} & \text{Hom}_{\mathcal{Z}(X), X_0} \times \text{Hom}_{\mathcal{Z}(X), X_1}
 \end{array}$$

where s_* and $(d_0, d_1)_*$ are maps induced by the composition by s and (d_0, d_1) , and s^* , d_0^* , d_1^* are maps induced by the precomposition by s , d_0 and d_1 . The projection $pr : \text{End}_{\mathcal{Z}(X)} \rightarrow \text{End}_{\mathcal{Z}(X)}$ is induced by the inclusion of diagrams $\{Z(X)\} \hookrightarrow \{X_0 \xleftarrow{\sim} Z(X) \xrightarrow{\sim} X_1\}$ (see [7], section 8). The idea is to define a P -modified endomorphism prop $\text{End}_{\mathcal{Y}(P)}$ with a form similar to that of $\text{End}_{\mathcal{Y}(X)}$, in order to get the prop morphism $\text{End}_{\mathcal{Y}(P)} \rightarrow \int_{X \in Ch_{\mathbb{K}}^P} \text{End}_{\mathcal{Y}(X)}$ induced by the morphisms $P \rightarrow \text{End}_X$, $X \in Ch_{\mathbb{K}}^P$. For this aim we use two pullbacks similar to those above with P -modified endomorphisms props and Σ -biobjects replacing the usual ones.

In our second step, we show that π is an acyclic fibration in \mathcal{P} in order to obtain the desired lifting $P \rightarrow \int_{X \in Ch_{\mathbb{K}}^P} \text{End}_{\mathcal{Y}(X)}$, which respects the P -algebra structures on the diagrams $\mathcal{V}(X)$ for every $X \in Ch_{\mathbb{K}}^P$. It endows the category of P -algebras with a functorial path object. Finally, we prove theorem 2.1 in section 2.6, by using lifting properties in the category of props and providing the desired zigzag of natural transformations $\varphi^* \xleftarrow{\sim} Z \xrightarrow{\sim} \psi^*$. Then we show how to deduce theorem 0.1.

Remark 2.2. We can also wonder about the homotopy invariance of the classifying space up to Quillen equivalences. Let P be a prop in \mathcal{E}_1 . Let $F : \mathcal{E}_1 \rightleftarrows \mathcal{E}_2 : G$ be a symmetric monoidal adjunction. The prop $F(P)$ is defined by applying the functor F entrywise to P : the fact that F is symmetric monoidal ensures the preservation of the composition products of P , giving to $F(P)$ a prop structure. Lemma 7.1 of [14] says that the adjoint pair (F, G) induces an adjunction $\overline{F} : \mathcal{E}_1^P \rightleftarrows \mathcal{E}_2^{F(P)} : \overline{G}$. Now suppose that (F, G) forms a Quillen adjunction. By Brown's lemma, the functor F preserves weak equivalences between cofibrant objects and the functor G preserves weak equivalences between fibrant objects. If all the objects of \mathcal{E}_1 and \mathcal{E}_2 are fibrant and cofibrant, then the adjoint pair $(\overline{F}, \overline{G})$ restricts to an adjunction $\overline{F} : w(\mathcal{E}_1)^P \rightleftarrows w(\mathcal{E}_2)^{F(P)} : \overline{G}$ and thus gives rise to a homotopy equivalence $\mathcal{N}w(\mathcal{E}_1)^P \sim \mathcal{N}w(\mathcal{E}_2)^{F(P)}$.

2.2. The path object $Z(X) = Z \otimes X$. Recall that in the model category structure of $Ch_{\mathbb{K}}$, the fibrations are the degreewise surjections, the cofibrations the degreewise injections and the weak equivalences the morphisms inducing isomorphisms in homology. The category $Ch_{\mathbb{K}}$ has moreover the simplifying feature that the product and the coproduct coincide. Let Z be the chain complex defined by

$$Z = \mathbb{K}\rho_0 \oplus \mathbb{K}\rho_1 \oplus \mathbb{K}\sigma_0 \oplus \mathbb{K}\sigma_1 \oplus \mathbb{K}\tau.$$

The elements τ , ρ_0 and ρ_1 are three generators of degree 0 and σ_0 , σ_1 two generators of degree -1 . The differential d_Z is defined by $d_Z(\sigma_0) = d_Z(\sigma_1) = 0$, $d_Z(\tau) = 0$, $d_Z(\rho_0) = \sigma_0$ and $d_Z(\rho_1) = \sigma_1$.

Lemma 2.3. *The chain complex $Z \otimes X$ defines a path object on X in $Ch_{\mathbb{K}}$, fitting in a factorization $X \xrightarrow{\sim}_s Z \otimes X \xrightarrow{(d_0, d_1)} X \oplus X$ of the diagonal $\Delta = (id_X, id_X) : X \rightarrow X \oplus X$ such that s is an acyclic cofibration and (d_0, d_1) a fibration.*

Proof. Let $s : X \rightarrow Z \otimes X$ be the map defined by $s(x) = \tau \otimes x$. Given the differential of Z , the map s is clearly an injective morphism of $Ch_{\mathbb{K}}$, i.e a cofibration. We can also write $Z \otimes X \cong (\tilde{Z} \otimes X) \oplus X$ where

$$\tilde{Z} = \mathbb{K}\rho_0 \oplus \mathbb{K}\rho_1 \oplus \mathbb{K}\sigma_0 \oplus \mathbb{K}\sigma_1$$

is an acyclic complex. The acyclicity of \tilde{Z} implies that s is an acyclic cofibration. We now define a map $(d_0, d_1) : Z \otimes X \rightarrow X \oplus X$ such that $(d_0, d_1) \circ s = (id_X, id_X)$ and (d_0, d_1) is a fibration. The map d_0 is determined for every $x \in X$ by $d_0(\tau \otimes x) = x$ and $d_0(\sigma_0 \otimes x) = d_0(\sigma_1 \otimes x) = d_0(\rho_0 \otimes x) = d_0(\rho_1 \otimes x) = 0$. The map d_1 is determined for every $x \in X$ by $d_1(\rho_0 \otimes x) = x$, $d_1(\tau \otimes x) = x$, $d_1(\sigma_0 \otimes x) = d_1(\sigma_1 \otimes x) = d_1(\rho_1 \otimes x) = 0$. The map (d_0, d_1) is clearly a surjective chain complexes morphism, i.e a fibration, and satisfies the equality $(d_0, d_1) \circ s = (id_X, id_X)$. \square

The two advantages of this path object on X are its writing under the form of a tensor product with X and its decomposition in a direct sum of X with an acyclic complex.

2.3. The prop $End_{Z(P)}$. Consider the endomorphism prop of $Z(X)$:

$$\begin{aligned} End_{Z(X)}(m, n) &= Hom_{Ch_{\mathbb{K}}}(Z(X)^{\otimes m}, Z(X)^{\otimes n}) \\ &\cong Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m} \otimes X^{\otimes m}, Z^{\otimes n} \otimes X^{\otimes n}) \\ &\cong (Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes End_X(m, n). \end{aligned}$$

We define a P -modified endomorphism prop such that

$$\begin{aligned} End_{Z(P)}(m, n) &= (Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes P(m, n) \\ &= \bigoplus t_1^* \otimes \dots \otimes t_m^* \otimes t_1 \otimes \dots \otimes t_n \otimes P(m, n), \end{aligned}$$

where $t_i \in \{\rho_0, \rho_1, \sigma_0, \sigma_1, \tau\}$, together with the following structure maps:

-Vertical composition product. Let

$$\alpha \in t_1^* \otimes \dots \otimes t_k^* \otimes t_1 \otimes \dots \otimes t_n \otimes P(k, n)$$

and

$$\beta \in u_1^* \otimes \dots \otimes u_m^* \otimes u_1 \otimes \dots \otimes u_k \otimes P(m, k).$$

We set

$$\alpha \circ_v \beta = \begin{cases} \alpha \circ_v^P \beta & \text{if } (u_1, \dots, u_k) = (t_1, \dots, t_k), \\ 0 & \text{otherwise,} \end{cases}$$

where \circ_v^P is the vertical composition product of P .

-Horizontal product. Let

$$\alpha \in t_1^* \otimes \dots \otimes t_{m_1}^* \otimes t_1 \otimes \dots \otimes t_{n_1} \otimes P(m_1, n_1)$$

and

$$\beta \in u_1^* \otimes \dots \otimes u_{m_2}^* \otimes u_1 \otimes \dots \otimes u_{n_2} \otimes P(m_2, n_2).$$

We set

$$\begin{aligned} \alpha \circ_h \beta &= t_1^* \otimes \dots \otimes t_{m_1}^* \otimes u_1^* \otimes \dots \otimes u_{m_2}^* \\ &\quad \otimes t_1 \otimes \dots \otimes t_{n_1} \otimes u_1 \otimes \dots \otimes u_{n_2} \otimes (\alpha \mid_{P(m_1, n_1)} \circ_h^P \beta \mid_{P(m_2, n_2)}) \\ &\in t_1^* \otimes \dots \otimes t_{m_1}^* \otimes u_1^* \otimes \dots \otimes u_{m_2}^* \\ &\quad \otimes t_1 \otimes \dots \otimes t_{n_1} \otimes u_1 \otimes \dots \otimes u_{n_2} \otimes P(m_1 + n_1, m_2 + n_2), \end{aligned}$$

where \circ_h^P is the horizontal product of P .

-Actions of the symmetric groups. Let $\alpha = t_1^* \otimes \dots \otimes t_m^* \otimes t_1 \otimes \dots \otimes t_n \otimes \alpha_P \in \text{End}_{Z(P)}(m, n)$ with $\alpha_P \in P(m, n)$. The action of a permutation $\sigma \in \Sigma_m$ on the right of this prop element is given by $\alpha \cdot \sigma = t_{\sigma(1)}^* \otimes \dots \otimes t_{\sigma(m)}^* \otimes t_1 \otimes \dots \otimes t_n \otimes \alpha_P \cdot \sigma$. The action of a permutation $\tau \in \Sigma_n$ on the left of this prop element is given by $\tau \cdot \alpha = t_1^* \otimes \dots \otimes t_m^* \otimes t_{\tau^{-1}(1)} \otimes \dots \otimes t_{\tau^{-1}(n)} \otimes \tau \cdot \alpha_P$.

Let $X \in Ch_{\mathbb{K}}^P$ be a P -algebra. From the definition of $\text{End}_{Z(P)}(m, n)$, we easily see that the prop morphism $P \rightarrow \text{End}_X$ induces a prop morphism

$$\text{End}_{Z(P)} \rightarrow \int_{X \in Ch_{\mathbb{K}}^P} \text{End}_{Z(X)}.$$

2.4. The prop $\text{End}_{Z(P)}$.

2.4.1. *The pullback defining $\text{End}_{Z(X)}$ and its explicit maps.* For every $(m, n) \in \mathbb{N}^2$, we have a pullback

$$\begin{array}{ccc} \text{End}_{Z(X)}(m, n) & \longrightarrow & \text{End}_{X_0}(m, n) \oplus \text{End}_{X_1}(m, n) \\ \downarrow & & \downarrow (d_0^{\otimes m})^* \oplus (d_1^{\otimes m})^* \\ \text{End}_{Z(X)}(m, n) & \xrightarrow{(d_0^{\otimes n}, d_1^{\otimes n})_*} & \text{Hom}_{Z(X), X_0}(m, n) \oplus \text{Hom}_{Z(X), X_1}(m, n) \end{array}.$$

For every $X \in Ch_{\mathbb{K}}^P$ and $(m, n) \in \mathbb{N}^2$ we have the isomorphisms

$$\begin{aligned} \text{Hom}_{X, Z(X)}(m, n) &= \text{Hom}_{Ch_{\mathbb{K}}}(X^{\otimes m}, Z(X)^{\otimes n}) \\ &\cong \text{Hom}_{Ch_{\mathbb{K}}}(X^{\otimes m}, Z^{\otimes n} \otimes X^{\otimes n}) \\ &\cong Z^{\otimes n} \otimes \text{End}_X(m, n) \end{aligned}$$

and

$$\begin{aligned} \text{Hom}_{Z(X), X_i}(m, n) &= \text{Hom}_{Ch_{\mathbb{K}}}(Z(X)^{\otimes m}, X^{\otimes n}) \\ &\cong \text{Hom}_{Ch_{\mathbb{K}}}(Z^{\otimes m} \otimes X^{\otimes m}, X^{\otimes n}) \\ &\cong (Z^{\otimes m})^* \otimes \text{End}_{X_i}(m, n). \end{aligned}$$

Applying these isomorphisms, we get a pullback

$$\begin{array}{ccc} \text{End}_{Z(X)}(m, n) & \xrightarrow{\quad\quad\quad} & \text{End}_{X_0}(m, n) \oplus \text{End}_{X_1}(m, n) \\ \downarrow & & \downarrow \overline{(d_0^{\otimes m})^* \oplus (d_1^{\otimes m})^*} \\ (Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes \text{End}_X(m, n) & \xrightarrow[\overline{(d_0^{\otimes n}, d_1^{\otimes n})^*}]{} & (Z^{\otimes m})^* \otimes \text{End}_{X_0}(m, n) \oplus (Z^{\otimes m})^* \otimes \text{End}_{X_1}(m, n) \end{array}.$$

We have to make explicit the maps $\overline{(d_0^{\otimes n}, d_1^{\otimes n})^*}$ and $\overline{(d_0^{\otimes m})^* \oplus (d_1^{\otimes m})^*}$ and replace $\text{End}_{X_0}(m, n)$, $\text{End}_{X_1}(m, n)$ and $\text{End}_X(m, n)$ by $P_0(m, n)$, $P_1(m, n)$ and $P(m, n)$ to obtain a P -modified endomorphism prop $\{\text{End}_{Z(P)}(m, n)\}_{(m, n) \in \mathbb{N}^2}$ acting naturally on $\text{End}_{Z(X)}(m, n)$, $X \in Ch_{\mathbb{K}}^P$. Then we apply the same method to build a P -modified endomorphism prop $\text{End}_{Y(P)}$ acting naturally on $\text{End}_{Y(X)}$, $X \in Ch_{\mathbb{K}}^P$.

Lemma 2.4. *Let $\{\underline{z}_i\}_{i \in I}$ be a basis of $Z^{\otimes m}$. The map*

$$\overline{(d_1^{\otimes m})^*} : \text{End}_X(m, n) \rightarrow (Z^{\otimes m})^* \otimes \text{End}_X(m, n)$$

is defined by the formula

$$\overline{(d_1^{\otimes m})^*}(\xi) = \sum_{j \in J} (\underline{z}_j^* \otimes \xi) = \left(\sum_{j \in J} \underline{z}_j^* \right) \otimes \xi,$$

where J is the subset of I such that $d_1^{\otimes m}(\underline{z}_j \otimes \underline{x}) = \underline{x}$ for $\underline{x} \in X^{\otimes m}$ and $j \in J$.

Proof. First we give an explicit inverse to the well known isomorphism

$$\begin{aligned} \lambda : U^* \otimes \text{Hom}_{Ch_{\mathbb{K}}}(V, V') &\xrightarrow{\cong} \text{Hom}_{Ch_{\mathbb{K}}}(U \otimes V, V') \\ \varphi \otimes f &\mapsto [u \otimes v \mapsto \varphi(u).f(v)] \end{aligned}$$

where U is supposed to be of finite dimension. Let $\{u_i\}_i \in I$ be a basis of U . We have $\lambda = \sum_{i \in I} \lambda_i$ where

$$\begin{aligned} \lambda_i : \mathbb{K}u_i^* \otimes \text{Hom}_{Ch_{\mathbb{K}}}(V, V') &\rightarrow \text{Hom}_{Ch_{\mathbb{K}}}(\mathbb{K}u_i \otimes V, V') \\ u_i^* \otimes f &\mapsto u_i^*.f : u_i \otimes v \mapsto u_i^*(u_i).f(v) = f(v) \end{aligned}$$

so

$$\begin{aligned} \lambda^{-1} : \text{Hom}_{Ch_{\mathbb{K}}}(U \otimes V, V') &\rightarrow U^* \otimes \text{Hom}_{Ch_{\mathbb{K}}}(V, V') \\ f &\mapsto \sum_{i \in I} (u_i^* \otimes f|_{\mathbb{K}u_i \otimes V}). \end{aligned}$$

Let $\sigma : Z^{\otimes m} \otimes X^{\otimes m} \rightarrow (Z \otimes X)^{\otimes m}$ be the map permuting the variables. Recall that the map d_1 is determined for every $x \in X$ by $d_1(\rho_0 \otimes x) = x$, $d_1(\tau \otimes x) = x$, $d_1(\sigma_0 \otimes x) = d_1(\sigma_1 \otimes x) = d_1(\rho_1 \otimes x) = 0$. The map

$$\overline{(d_1^{\otimes m})^*} : \text{Hom}_{Ch_{\mathbb{K}}}(X^{\otimes m}, X^{\otimes n}) \rightarrow \text{Hom}_{Ch_{\mathbb{K}}}(Z^{\otimes m} \otimes X^{\otimes m}, X^{\otimes n}) \xrightarrow{\cong} (Z^{\otimes m})^* \otimes \text{Hom}_{Ch_{\mathbb{K}}}(X^{\otimes m}, X^{\otimes n})$$

is defined by

$$\xi \mapsto \xi \circ d_1^{\otimes m} \circ \sigma \mapsto \sum_{i \in I} (\underline{z}_i^* \otimes (\xi \circ d_1^{\otimes m} \circ \sigma) |_{\mathbb{K}_{\underline{z}_i} \otimes V}).$$

We obtain finally

$$\begin{aligned} \overline{(d_1^{\otimes m})^*} : \text{End}_X(m, n) &\rightarrow (Z^{\otimes m})^* \otimes \text{End}_X(m, n) \\ \xi &\mapsto \sum_{j \in J} (\underline{z}_j^* \otimes \xi) = \left(\sum_{j \in J} \underline{z}_j^* \right) \otimes \xi \end{aligned}$$

where J is the subset of I such that $d_1^{\otimes m}(\underline{z}_j \otimes \underline{x}) = \underline{x}$ for $\underline{x} \in X^{\otimes m}$ and $j \in J$. If $j \notin J$ then $d_1^{\otimes m} |_{\mathbb{K}_{\underline{z}_j} \otimes X^{\otimes m}} = 0$. \square

Recall that the map $d_0 : Z \otimes X \rightarrow X$ is defined for every $x \in X$ by $d_0(\tau \otimes x) = x$ and $d_0(\sigma_0 \otimes x) = d_0(\sigma_1 \otimes x) = d_0(\rho_0 \otimes x) = d_0(\rho_1 \otimes x) = 0$. As previously, the map $\overline{(d_0^{\otimes m})^*}$ has a form similar to that of $\overline{(d_1^{\otimes m})^*}$, and we have determined $\overline{(d_0^{\otimes m})^*} \oplus \overline{(d_1^{\otimes m})^*}$.

Lemma 2.5. *The map $\overline{(d_0^{\otimes n}, d_1^{\otimes n})^*}$ is determined by*

$$\overline{(d_0^{\otimes n}, d_1^{\otimes n})^*} : \underline{z}_j^* \otimes \underline{z}_i' \otimes \xi \mapsto \sum_{k \in I} (\underline{z}_k^* \otimes ((d_0^{\otimes n}, d_1^{\otimes n}) \circ \underline{z}_j^*(-) \cdot \underline{z}_i' \otimes \xi) |_{\mathbb{K}_{\underline{z}_k} \otimes X^{\otimes m}}).$$

Proof. Let $\{\underline{z}_i'\}_{i \in I'}$ be the basis of $Z^{\otimes n}$. We have the isomorphism

$$\begin{aligned} (Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes \text{Hom}_{Ch_{\mathbb{K}}}(X^{\otimes m}, X^{\otimes n}) &\rightarrow \text{Hom}_{Ch_{\mathbb{K}}}(Z^{\otimes m} \otimes X^{\otimes m}, Z^{\otimes n} \otimes X^{\otimes n}) \\ \underline{z}_j^* \otimes \underline{z}_i' \otimes \xi &\mapsto \underline{z}_j^*(-) \cdot \underline{z}_i' \otimes \xi \end{aligned}$$

that we compose with

$$\begin{aligned} (d_0^{\otimes n}, d_1^{\otimes n}) : Z^{\otimes n} \otimes X^{\otimes n} &\rightarrow X_0^{\otimes n} \oplus X_1^{\otimes n} \\ \underline{z}_j \otimes x &\mapsto \begin{cases} x \oplus x & \text{if } j \in J', \\ x \oplus 0 \text{ or } 0 \oplus x & \text{otherwise,} \end{cases} \end{aligned}$$

where J' is the subset of I such that $d_0 |_{\mathbb{K}_{\underline{z}_j} \otimes X^{\otimes n}} \neq 0$ and $d_1 |_{\mathbb{K}_{\underline{z}_j} \otimes X^{\otimes n}} \neq 0$ for $j \in J'$. Finally we compose with the isomorphism

$$\begin{aligned} \text{Hom}_{Ch_{\mathbb{K}}}(Z^{\otimes m} \otimes X^{\otimes m}, X_0^{\otimes n} \oplus X_1^{\otimes n}) &\xrightarrow{\cong} (Z^{\otimes m})^* \otimes \text{Hom}_{Ch_{\mathbb{K}}}(X^{\otimes m}, X_0^{\otimes n} \oplus X_1^{\otimes n}) \\ f &\mapsto \sum_{i \in I} (\underline{z}_i^* \otimes f |_{\mathbb{K}_{\underline{z}_i} \otimes X^{\otimes m}}) \end{aligned}$$

and get the map

$$\overline{(d_0^{\otimes n}, d_1^{\otimes n})^*} : \underline{z}_j^* \otimes \underline{z}_i' \otimes \xi \mapsto \sum_{k \in I} (\underline{z}_k^* \otimes ((d_0^{\otimes n}, d_1^{\otimes n}) \circ \underline{z}_j^*(-) \cdot \underline{z}_i' \otimes \xi) |_{\mathbb{K}_{\underline{z}_k} \otimes X^{\otimes m}}).$$

\square

2.4.2. The associated P -modified prop. The key observation is that these two maps $\overline{(d_0^{\otimes m})^*} \oplus \overline{(d_1^{\otimes m})^*}$ and $\overline{(d_0^{\otimes n}, d_1^{\otimes n})_*}$, fixing the prop structure on $End_{\mathcal{Z}(X)}(m, n)$ in function of those of $(Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes End_X(m, n)$ and $End_{X_0}(m, n) \oplus End_{X_1}(m, n)$, do not modify the operations $\xi \in End_X(m, n)$ themselves. Therefore, we replace $End_{X_0}(m, n)$, $End_{X_1}(m, n)$ and $End_X(m, n)$ by $P_0(m, n)$, $P_1(m, n)$ and $P(m, n)$ to get this new pullback

$$\begin{array}{ccc} End_{\mathcal{Z}(P)}(m, n) & \longrightarrow & P_0(m, n) \oplus P_1(m, n) \\ \downarrow & & \downarrow \overline{(d_0^{\otimes m})^*} \oplus \overline{(d_1^{\otimes m})^*} \\ (Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes P(m, n) & \xrightarrow[\overline{(d_0^{\otimes n}, d_1^{\otimes n})_*}]{} & (Z^{\otimes m})^* \otimes P_0(m, n) \oplus (Z^{\otimes m})^* \otimes P_1(m, n) \end{array} .$$

The explicit formulae of the applications defining this pullback, given by lemmas 2.4 and 2.5, show that these replacements do not break the prop structure transfer. Thus we get the desired P -modified endomorphism prop $End_{\mathcal{Z}(P)}$ having the same shape as that of $End_{\mathcal{Z}(X)}$ and thus acting naturally on the associated diagram of P -algebras:

$$End_{\mathcal{Z}(P)} \rightarrow \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{Z}(X)}.$$

2.5. The prop $End_{\mathcal{Y}(P)}$ and the functorial path object in P -algebras. Now let us define $End_{\mathcal{Y}(P)}$. For every $(m, n) \in \mathbb{N}^2$, the pullback

$$\begin{array}{ccc} End_{\mathcal{Y}(X)}(m, n) & \longrightarrow & End_{\mathcal{Z}(X)}(m, n) \\ \downarrow & & \downarrow (s^{\otimes m})^* \circ pr \\ End_X(m, n) & \xrightarrow[\overline{(s^{\otimes n})_*}]{} & Hom_{X, \mathcal{Z}(X)}(m, n) \end{array}$$

induces via the isomorphisms explained at the beginning of 3.3 and 3.4.1 a pullback

$$\begin{array}{ccc} End_{\mathcal{Y}(X)}(m, n) & \longrightarrow & End_{\mathcal{Z}(X)}(m, n) \\ \downarrow & & \downarrow \overline{(s^{\otimes m})^* \circ pr} \\ End_X(m, n) & \xrightarrow[\overline{(s^{\otimes n})_*}]{} & Z^{\otimes n} \otimes End_X(m, n) \end{array} .$$

In the same manner as before, given that $s : X \rightarrow Z \otimes X$ sends every $x \in X$ to $\tau \otimes x$, the map $\overline{(s^{\otimes m})^*}$ is of the form

$$(Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes End_X(m, n) \rightarrow Z^{\otimes n} \otimes End_X(m, n)$$

$$\underline{z}_j^* \otimes \underline{z}'_i \otimes \xi \mapsto \begin{cases} \underline{z}'_i \otimes \xi & \text{if } j \in K, \\ 0 & \text{otherwise,} \end{cases}$$

where K is a certain subset of I and $\overline{(s^{\otimes n})_*}$ is of the form

$$End_X(m, n) \rightarrow Z^{\otimes n} \otimes End_X(m, n)$$

$$\xi \mapsto \sum_{i \in K'} \underline{z}'_i \otimes \xi$$

where K' is a certain subset of I' . These two maps $\overline{(s^{\otimes m})^* \circ pr}$ and $\overline{(s^{\otimes n})_*}$, fixing the prop structure on $End_{Y(X)}(m, n)$ in function of those of $End_{Z(X)}(m, n)$ and $End_X(m, n)$, do not modify the operations $\xi \in End_X(m, n)$ themselves. Therefore, we replace $End_X(m, n)$ by $P(m, n)$ and $End_{Z(X)}(m, n)$ by $End_{Z(P)}(m, n)$ to get this new pullback

$$\begin{array}{ccc} End_{Y(P)}(m, n) & \longrightarrow & End_{Z(P)}(m, n) \\ \downarrow & & \downarrow \overline{(s^{\otimes m})^* \circ pr} \\ P(m, n) & \xrightarrow[\overline{(s^{\otimes n})_*}]{} & Z^{\otimes n} \otimes P(m, n) \end{array} .$$

The explicit formulae of the applications defining this pullback show that these replacements do not break the prop structure transfer. Thus we get the desired P -modified endomorphism prop $End_{Y(P)}$ having the same shape as that of $End_{Y(X)}$ and thus acting naturally on the associated diagram of P -algebras:

$$End_{Y(P)} \rightarrow \int_{X \in Ch_{\mathbb{K}}^P} End_{Y(X)}.$$

We finally obtain the following lemma:

Lemma 2.6. *There is a commutative diagram of props*

$$\begin{array}{ccccc} & & End_{Y(P)} & \longrightarrow & \int_{X \in Ch_{\mathbb{K}}^P} End_{Y(X)} \\ & & \downarrow \pi & & \downarrow \\ P & \xrightarrow{=} & P & \longrightarrow & \int_{X \in Ch_{\mathbb{K}}^P} End_{Y(X)} \end{array}$$

Now we want to prove that the morphism $P \rightarrow \int_{X \in Ch_{\mathbb{K}}^P} End_{Y(X)}$ lifts to a morphism $P \rightarrow \int_{X \in Ch_{\mathbb{K}}^P} End_{Y(X)}$:

Lemma 2.7. *The map π is an acyclic fibration in the category of props.*

Proof. According to the model category structure on \mathcal{P} , it is sufficient to prove that for every $(m, n) \in \mathbb{N}^2$, $\pi(m, n)$ is an acyclic fibration of chain complexes. The map $\pi(m, n)$ is given by the base extension

$$\pi(m, n) = P(m, n) \underset{Hom_{P, Z(P)}(m, n)}{\times} \phi(m, n) \underset{Hom_{Z(P), P_0}(m, n) \oplus Hom_{Z(P), P_1}(m, n)}{\times} (P_0(m, n) \oplus P_1(m, n))$$

where

$$\phi(m, n) : End_{Z(P)}(m, n) \rightarrow Hom_{P, Z(P)}(m, n) \underset{P_0(m, n) \oplus P_1(m, n)}{\times} (Hom_{Z(P), P_0}(m, n) \oplus Hom_{Z(P), P_1}(m, n))$$

comes from the diagram

$$\begin{array}{ccc}
 \text{End}_{Z(P)}(m, n) & \xrightarrow{\overline{(d_0^{\otimes n}, d_1^{\otimes n})_*}} & \\
 \searrow \phi(m, n) & \downarrow \text{pullback} & \searrow \\
 & \text{Hom}_{Z(P), P_0}(m, n) \oplus \text{Hom}_{Z(P), P_1}(m, n) & \\
 \downarrow \overline{(s^{\otimes m})^*} & \downarrow \overline{(s^{\otimes m})^*} \oplus \overline{(s^{\otimes m})^*} & \\
 \text{Hom}_{P, Z(P)}(m, n) & \xrightarrow{\overline{(d_0^{\otimes n}, d_1^{\otimes n})_*}} & P_0(m, n) \oplus P_1(m, n)
 \end{array}$$

i.e

$$\begin{array}{ccc}
 (Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes P(m, n) & \xrightarrow{\overline{(d_0^{\otimes n}, d_1^{\otimes n})_*}} & \\
 \searrow \phi(m, n) & \downarrow \text{pullback} & \searrow \\
 & (Z^{\otimes m})^* \otimes (P_0(m, n) \oplus P_1(m, n)) & \\
 \downarrow \overline{(s^{\otimes m})^*} & \downarrow \overline{(s^{\otimes m})^*} \oplus \overline{(s^{\otimes m})^*} & \\
 Z^{\otimes n} \otimes P(m, n) & \xrightarrow{\overline{(d_0^{\otimes n}, d_1^{\otimes n})_*}} & P_0(m, n) \oplus P_1(m, n)
 \end{array}$$

We have an isomorphism

$$\begin{aligned}
 P_0(m, n) \oplus P_1(m, n) &\xrightarrow{\cong} (\mathbb{K}p_0 \oplus \mathbb{K}p_1) \otimes P(m, n) \\
 p \oplus p' &\mapsto p_0 \otimes p + p_1 \otimes p
 \end{aligned}$$

where p_0 and p_1 are two generators of degree 0. The previous computations give

$$\begin{aligned}
 \overline{(d_0^{\otimes n}, d_1^{\otimes n})_*} : Z^{\otimes n} \otimes P(m, n) &\rightarrow (\mathbb{K}p_0 \oplus \mathbb{K}p_1) \otimes P(m, n) \\
 \underline{z}'_i \otimes p &\mapsto \begin{cases} (p_0 \oplus p_1) \otimes p & \text{if } i \in J', \\ p_0 \otimes p \text{ or } p_1 \otimes p & \text{otherwise,} \end{cases}
 \end{aligned}$$

and the map

$$\overline{(s^{\otimes m})^*} \oplus \overline{(s^{\otimes m})^*} : (Z^{\otimes m})^* \otimes (\mathbb{K}p_0 \oplus \mathbb{K}p_1) \otimes P(m, n) \rightarrow (\mathbb{K}p_0 \oplus \mathbb{K}p_1) \otimes P(m, n)$$

is defined by

$$\underline{z}_j^* \otimes (\lambda p_0 \oplus \mu p_1) \otimes p \mapsto \begin{cases} (\lambda p_0 \oplus \mu p_1) \otimes p \text{ or } \lambda p_0 \otimes p \text{ or } \mu p_1 \otimes p, & \text{if } j \in K, \\ 0 = 0 \otimes p & \text{otherwise.} \end{cases}$$

We have similar results for the two maps starting from $(Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes P(m, n)$. We deduce that the previous diagram is the image under the functor $- \otimes P(m, n)$ of the dual pushout-product

$$\begin{array}{ccc}
 Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}, Z^{\otimes n}) & \xrightarrow{(g_{d_0, d_1})_*} & Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}, \mathbb{K}p_0 \oplus \mathbb{K}p_1) \\
 \searrow (f_s^*, (g_{d_0, d_1})_*) & \text{pullback} & \downarrow f_s^* \\
 Hom_{Ch_{\mathbb{K}}}(\mathbb{K}, Z^{\otimes n}) & \xrightarrow{(g_{d_0, d_1})_*} & Hom_{Ch_{\mathbb{K}}}(\mathbb{K}, \mathbb{K}p_0 \oplus \mathbb{K}p_1)
 \end{array}$$

f_s^* (from $Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}, Z^{\otimes n})$ to $Hom_{Ch_{\mathbb{K}}}(\mathbb{K}, Z^{\otimes n})$)

modulo the isomorphisms

$$Z^{\otimes n} \cong Hom_{Ch_{\mathbb{K}}}(\mathbb{K}, Z^{\otimes n}),$$

$$(Z^{\otimes m})^* \otimes Z^{\otimes n} \cong Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}, Z^{\otimes n}),$$

$$(Z^{\otimes m})^* \otimes (\mathbb{K}p_0 \oplus \mathbb{K}p_1) \cong Hom_{Ch_{\mathbb{K}}}(Z^{\otimes m}, \mathbb{K}p_0 \oplus \mathbb{K}p_1)$$

and

$$\mathbb{K}p_0 \oplus \mathbb{K}p_1 \cong Hom_{Ch_{\mathbb{K}}}(\mathbb{K}, \mathbb{K}p_0 \oplus \mathbb{K}p_1).$$

The map $g_{d_0, d_1} : Z^{\otimes n} \rightarrow \mathbb{K}p_0 \oplus \mathbb{K}p_1$ is surjective so it is a fibration of chain complexes. Recall that we have a decomposition of Z into $Z = \tilde{Z} \oplus \mathbb{K}\tau$ where \tilde{Z} is acyclic, which implies a decomposition of $Z^{\otimes m}$ of the form $Z^{\otimes m} \cong S_m \oplus \mathbb{K}(\tau^{\otimes n})$ where S_m is acyclic because it is a sum of tensor products containing \tilde{Z} . The map f_s is an injection sending \mathbb{K} on $\mathbb{K}(\tau^{\otimes n})$ so it is a cofibration, and S_m is acyclic so f_s is an acyclic cofibration. Applying the axiom MM1' in $Ch_{\mathbb{K}}$ we conclude that $(f_s^*, (g_{d_0, d_1})_*)$ is an acyclic fibration. Therefore $\phi(m, n) = (f_s^*, (g_{d_0, d_1})_*) \otimes id_{P(m, n)}$ is an acyclic fibration, and so is $\pi(m, n)$, given that the class of acyclic fibrations is stable by base extension. \square

We have proved the following result:

Proposition 2.8. *There is a prop morphism $P \rightarrow \int_{X \in Ch_{\mathbb{K}}^P} End_{\mathcal{Y}(X)}$, and consequently a functorial path object $Z : (Ch_{\mathbb{K}})^P \rightarrow (Ch_{\mathbb{K}})^P$ in the category of cofibrant-fibrant P -algebras $(Ch_{\mathbb{K}})^P$.*

2.6. Proof of the final result. Consider now the square of inclusions of diagrams

$$\begin{array}{ccc}
 \mathcal{T}(X) & \xrightarrow{t} & \mathcal{V}(X) \\
 \downarrow u & & \downarrow v \\
 \mathcal{Z}(X) & \xrightarrow{w} & \mathcal{Y}(X)
 \end{array}$$

where $\mathcal{V}(X)$, $\mathcal{Z}(X)$ and $\mathcal{Y}(X)$ are the diagrams defined previously and $\mathcal{T}(X)$ is the diagram $\{X_0, X_1\}$ consisting of two copies of X and no arrows between them. This square of inclusions induces the following commutative square of endomorphisms props

$$\begin{array}{ccc} \text{End}_{\mathcal{Y}(X)} & \xrightarrow{w_*} & \text{End}_{\mathcal{Z}(X)} \\ v_* \downarrow & & \downarrow u^* \\ \text{End}_{\mathcal{V}(X)} & \xrightarrow{t^*} & \text{End}_{\mathcal{T}(X)} \end{array}$$

where u^* , v^* , t^* and w^* are the maps induced by the inclusions of the associated diagrams P -algebras. We have a commutative diagram of P -modified endomorphism props reflecting this square

$$\begin{array}{ccc} \text{End}_{\mathcal{Y}(P)} & \xrightarrow{w_*} & \text{End}_{\mathcal{Z}(P)} \\ v_* \downarrow & & \downarrow u^* \\ \text{End}_{\mathcal{V}(P)} = P & \xrightarrow{t^*} & \text{End}_{\mathcal{T}(P)} = P_0 \times P_1 \end{array}$$

where v^* is the acyclic fibration π of lemma 2.7 and u^* is a fibration because it is clearly surjective in each biarity (recall that the surjective morphisms are the fibrations of $Ch_{\mathbb{K}}$ and that the fibrations of \mathcal{P} are determined componentwise). Now we can use this commutative square to prove the final result:

Theorem 2.9. *Let P be a cofibrant prop and $\varphi, \psi : P \rightarrow P$ two homotopic prop morphisms, then there exists a diagram of functors*

$$\varphi^* \xleftarrow{\sim} Z \xrightarrow{\sim} \psi^*$$

where Z is the path object functor defined in proposition 2.8 and the natural transformations are pointwise acyclic fibrations.

Proof. We consider a cylinder object of P fitting in a diagram of the form:

$$P \vee P \xrightarrow{(d_0, d_1)} \tilde{P} \xrightarrow{s_0} P$$

The components d_0 and d_1 of the morphism (d_0, d_1) are acyclic cofibrations because P is cofibrant by assumption (see lemma 4.4 in [4]) and s_0 an acyclic fibration. The fact that φ and ψ are homotopic implies the existence of a lifting in

$$\begin{array}{ccc} P \vee P & \xrightarrow{(\varphi, \psi)} & P \\ (d_0, d_1) \downarrow & \nearrow h & \downarrow \\ \tilde{P} & \longrightarrow & 0 \end{array}$$

We produce the lifting

$$\begin{array}{ccc} I & \longrightarrow & \text{End}_{\mathcal{Y}(P)} \\ \downarrow & \nearrow k & \downarrow \sim v^* \\ P & \xrightarrow{\varphi} & P \end{array}$$

(by the axiom MC4 of model categories, see [4]) and form $(\varphi \circ s_0, h) : \tilde{P} \rightarrow P_0 \times P_1$ in order to get the following commutative diagram:

$$\begin{array}{ccccc}
 P & \xrightarrow{k} & \text{End}_{\mathcal{Y}(P)} & \xrightarrow{w^*} & \text{End}_{\mathcal{Z}(P)} \\
 \downarrow d_0 \sim & & \nearrow l & & \downarrow u^* \\
 \tilde{P} & \xrightarrow{(\varphi \circ s_0, h)} & P_0 \times P_1 & &
 \end{array}$$

We have $(\varphi \circ s_0, h) \circ d_0 = (\varphi \circ s_0 \circ d_0, h \circ d_0) = (\varphi, \varphi)$ and $u^* \circ w^* \circ k = t^* \circ v^* \circ k = t^* \circ \varphi = (\varphi, \varphi)$ so this diagram is indeed commutative and there exists a lifting (axiom MC4) $l : \tilde{P} \rightarrow \text{End}_{\mathcal{Z}(P)}$. Then we form $l \circ d_1 : P \rightarrow \text{End}_{\mathcal{Z}(P)}$ and observe that $u^* \circ l \circ d_1 = (\varphi \circ s_0, h) \circ d_1 = (\varphi \circ s_0 \circ d_1, h \circ d_1) = (\varphi, \psi)$, i.e we obtain the following diagram:

$$\begin{array}{ccccc}
 & \text{End}_{\mathcal{Z}(P)} & \longrightarrow & \int_{X \in \text{Ch}_{\mathbb{K}}^P} \text{End}_{\mathcal{Z}(X)} & \\
 \nearrow l \circ d_1 & \downarrow u^* & & \downarrow & \\
 P & \xrightarrow{(\varphi, \psi)} P_0 \times P_1 & \longrightarrow & \int_{X \in \text{Ch}_{\mathbb{K}}^P} \text{End}_{\mathcal{T}(X)} &
 \end{array}$$

and consequently a diagram of functors $\varphi^* \xleftarrow{\sim} Z \xrightarrow{\sim} \psi^*$. The functorial path object Z on $\text{Ch}_{\mathbb{K}}$ preserves weak equivalences and restrict to an endofunctor of $w\text{Ch}_{\mathbb{K}}$, so the associated functorial path object Z on $\text{Ch}_{\mathbb{K}}^P$ do the same. Moreover, the natural transformations are weak equivalences in each component, so this diagram restricts to the desired diagram of endofunctors of $w\text{Ch}_{\mathbb{K}}^P$. \square

Now we can conclude the proof of theorem 0.1 in the case $\mathcal{E} = \text{Ch}_K$:

Theorem 2.10. *Let $\text{Ch}_{\mathbb{K}}$ be the category of \mathbb{Z} -graded chain complexes over a field \mathbb{K} of characteristic zero. Let $\varphi : P \xrightarrow{\sim} Q$ be a weak equivalence between two cofibrant props. The map φ gives rise to a functor $\varphi^* : w(\text{Ch}_{\mathbb{K}})^Q \rightarrow w(\text{Ch}_{\mathbb{K}})^P$ which induces a weak equivalence of simplicial sets $\mathcal{N}\varphi^* : \mathcal{N}w(\text{Ch}_{\mathbb{K}})^Q \xrightarrow{\sim} \mathcal{N}w(\text{Ch}_{\mathbb{K}})^P$.*

Proof. Recall that \mathcal{P} is the category of props in Ch_K . Let us suppose first that $\varphi : P \xrightarrow{\sim} Q$ is an acyclic cofibration between two cofibrant props of \mathcal{P} . All objects in $\text{Ch}_{\mathbb{K}}$ are fibrant, so by definition of the model category structure on \mathcal{P} the prop P is fibrant and thus we have the following lifting

$$\begin{array}{ccc}
 P & \xrightarrow{=} & P \\
 \downarrow \varphi & \nearrow s & \downarrow \\
 Q & \longrightarrow & pt
 \end{array}$$

where $s : Q \xrightarrow{\sim} P$ satisfies

$$\begin{cases} s \circ \varphi = id_P \\ \varphi \circ s \sim id_Q \end{cases}$$

(the relation \sim is the homotopy relation for the model category structure of \mathcal{P}). These maps induce functors $\varphi^* : (w\mathcal{E}^{cf})^Q \rightarrow (w\mathcal{E}^{cf})^P$ and $s^* : (w\mathcal{E}^{cf})^P \rightarrow (w\mathcal{E}^{cf})^Q$. Applying the simplicial nerve functor, we obtain

$$\begin{cases} \mathcal{N}(s \circ \varphi)^* = \mathcal{N}\varphi^* \circ \mathcal{N}s^* = id_{(w\mathcal{E}^{cf})^P} \\ \mathcal{N}(\varphi \circ s)^* = \mathcal{N}s^* \circ \mathcal{N}\varphi^* \sim id_{(w\mathcal{E}^{cf})^Q} \end{cases}$$

so $\mathcal{N}\varphi^*$ is a homotopy equivalence in $sSet$, which implies that it is a weak equivalence of simplicial sets. The functor

$$\begin{aligned} \mathcal{P} &\rightarrow sSet \\ P &\mapsto \mathcal{N}w(\mathcal{E}^{cf})^P \end{aligned}$$

is defined between two model categories, and maps the acyclic cofibrations between cofibrant objects to weak equivalences, so it preserves weak equivalences between cofibrant objects according to Brown's lemma. \square

2.7. The general case of a category \mathcal{E} tensored over $Ch_{\mathbb{K}}$. To complete our results we explain how the proof of theorem 2.9 extends to a category \mathcal{E} tensored over $Ch_{\mathbb{K}}$.

Theorem 2.11. *Let \mathcal{E} be a cofibrantly generated symmetric monoidal model category over $Ch_{\mathbb{K}}$. Let $\varphi : P \xrightarrow{\sim} Q$ be a weak equivalence between two cofibrant props defined in $Ch_{\mathbb{K}}$. This morphism φ gives rise to a functor $\varphi^* : w(\mathcal{E}^c)^Q \rightarrow w(\mathcal{E}^c)^P$ which induces a weak equivalence of simplicial sets $\mathcal{N}\varphi^* : \mathcal{N}w(\mathcal{E}^c)^Q \rightarrow \mathcal{N}w(\mathcal{E}^c)^P$.*

Proof. The chain complex Z defined previously is itself the path object on C^0 , so we have the commutative diagram

$$\begin{array}{ccc} & & C^0 \\ & \nearrow \scriptstyle \sim & \nearrow \scriptstyle d_0 \\ C^0 & \xrightarrow{\scriptstyle s} & Z \\ & \searrow \scriptstyle \sim & \searrow \scriptstyle d_1 \\ & & C^0 \end{array}$$

Given that C^0 is the unit of $Ch_{\mathbb{K}}$, for any $X \in \mathcal{E}$ we have $C^0 \otimes X \cong X$, thus by applying the functor $- \otimes X$ we get the commutative diagram

$$\begin{array}{ccc} & & X_0 \\ & \nearrow \scriptstyle \sim & \nearrow \scriptstyle d_0 \otimes id_X \\ X & \xrightarrow{\scriptstyle s \otimes id_X} & Z \otimes X \\ & \searrow \scriptstyle \sim & \searrow \scriptstyle d_1 \otimes id_X \\ & & X_1 \end{array}$$

The axiom MM1 for the external tensor product \otimes implies that if X is cofibrant, then the functor $- \otimes X$ preserves acyclic cofibrations of $Ch_{\mathbb{K}}$ (all the objects of $Ch_{\mathbb{K}}$ are cofibrant) and thus, by Brown's lemma, it preserves the weak equivalences. Therefore $s \otimes id_X$ is still an acyclic cofibration and $d_0 \otimes id_X, d_1 \otimes id_X$ are weak equivalences. Moreover, given the properties of \otimes and the fact that endomorphism props in $Ch_{\mathbb{K}}$ for objects of \mathcal{E} are defined with the external hom bifunctor $Hom_{\mathcal{E}}(-, -)$ of \mathcal{E} , we have the following isomorphisms:

$$\begin{aligned} End_{Z \otimes X}(m, n) &= Hom_{\mathcal{E}}((Z \otimes X)^{\otimes m}, (Z \otimes X)^{\otimes n}) \\ &\cong Hom_{\mathcal{E}}(Z^{\otimes m} \otimes X^{\otimes m}, Z^{\otimes n} \otimes X^{\otimes n}) \\ &\cong (Z^{\otimes m})^* \otimes Z^{\otimes n} \otimes End_X(m, n) \end{aligned}$$

$$\begin{aligned} Hom_{X, Z \otimes X}(m, n) &= Hom_{\mathcal{E}}(X^{\otimes m}, (Z \otimes X)^{\otimes n}) \\ &\cong Hom_{\mathcal{E}}(X^{\otimes m}, Z^{\otimes n} \otimes X^{\otimes n}) \\ &\cong Z^{\otimes n} \otimes End_X(m, n), \end{aligned}$$

and

$$\begin{aligned} Hom_{Z \otimes X, X_i}(m, n) &= Hom_{\mathcal{E}}((Z \otimes X)^{\otimes m}, X^{\otimes n}) \\ &\cong Hom_{\mathcal{E}}(Z^{\otimes m} \otimes X^{\otimes m}, X^{\otimes n}) \\ &\cong (Z^{\otimes m})^* \otimes End_{X_i}(m, n). \end{aligned}$$

The proofs of 3.3, 3.4 and 3.5 extend without changes to the case of a category \mathcal{E} tensored over $Ch_{\mathbb{K}}$: we still work in $Ch_{\mathbb{K}}$, and as before the operations associated to $s \otimes id_X, d_0 \otimes id_X$ and $d_1 \otimes id_X$ in the pullbacks do not transform the elements of $End_X(m, n)$ themselves, so that the replacement of $End_X(m, n)$ by $P(m, n)$ does not break the transfert of prop structure in these pullbacks. We obtain a diagram of endofunctors $\varphi^* \xleftarrow{\sim} Z \xrightarrow{\sim} \psi^*$ of $(\mathcal{E}^c)^P$ where the natural transformations are weak equivalences in each component, so this diagram restricts to the desired diagram of endofunctors of $w(\mathcal{E}^c)^P$. The theorem 0.1 is proved in the general case. \square

3. THE SUBCATEGORY OF ACYCLIC FIBRATIONS

The goal of this section is to show that the classifying space $\mathcal{N}w(\mathcal{E}^{cf})^P$ is weakly equivalent to $\mathcal{N}fw(\mathcal{E}^{cf})^P$, that is the nerve of the subcategory of acyclic fibrations. It works in the broader context of a category \mathcal{E} tensored over any symmetric monoidal cofibrantly generated model category \mathcal{C} . The following result is a key point in the proof of theorem 0.2:

Proposition 3.1. *Let \mathcal{E} be a symmetric monoidal cofibrantly generated model category over \mathcal{C} satisfying the limit monoid axioms. Let P be a cofibrant prop with non-empty inputs (or outputs) defined in \mathcal{C} . The inclusion of categories $i : fw(\mathcal{E}^{cf})^P \hookrightarrow w(\mathcal{E}^{cf})^P$ gives rise to a weak equivalence of simplicial sets $\mathcal{N}fw(\mathcal{E}^{cf})^P \xrightarrow{\sim} \mathcal{N}w(\mathcal{E}^{cf})^P$.*

For this aim, we will have to deal in section 3.2 with the lifting of P -algebras structures in a certain diagram category. Therefore we need a compatibility between tensor and model structures on diagram categories, which we check in section 3.1.

3.1. The monoidal model structure of a diagram category. Let us start by recalling the model category structure on a diagram category:

Definition 3.2. Let \mathcal{M} be a cocomplete category.

- (1) Let X be an object of \mathcal{M} and S a set. We set $S \otimes X = \coprod_S X$.
- (2) Let I be a small category and X an object of \mathcal{M} . We consider an I -diagram $F_X^i : I \rightarrow \mathcal{M}$ defined by $F_X^i = \text{Mor}_I(i, -) \otimes X$, i.e for every $j \in I$, $F_X^i(j) = \coprod_{\text{Mor}_I(i, j)} X$.
- (3) Let \mathcal{K} be a set of morphisms of \mathcal{M} . We denote by $F_{\mathcal{K}}^I$ the set of morphisms in \mathcal{M}^I of the form $\text{Mor}_I(i, -) \otimes f : F_X^i \rightarrow F_Y^i$ where $f : A \rightarrow B \in \mathcal{K}$ and $i \in I$.

Theorem 3.3. (cf. [11], theorem 11.6.1) *Let I be a small category. Let \mathcal{M} be a cofibrantly generated model category, with C as set of generating cofibrations and C_a as set of generating acyclic cofibrations. The diagram category \mathcal{M}^I is endowed with a cofibrantly generated model category structure such that:*

- (1) *$f : X \xrightarrow{\sim} Y$ is a weak equivalence (respectively a fibration) of \mathcal{M}^I if and only if for every $i \in I$ the morphism $f(i) : X(i) \xrightarrow{\sim} Y(i)$ is a weak equivalence (respectively a fibration) of \mathcal{M} ;*
- (2) *The set of generating cofibrations (respectively acyclic generating cofibrations) of \mathcal{M}^I is F_C^I (respectively $F_{C_a}^I$).*

Consequently, a cofibration is a retract or a transfinite composition of pushouts of elements of F_C^I .

Proposition 3.4. (cf. [11], proposition 11.6.3) *If $f : M \rightarrow N$ is a cofibration of I -diagrams, then $f(i) : M(i) \rightarrow N(i)$ is a cofibration in \mathcal{M} for each $i \in I$.*

We now assume that \mathcal{M} is a cofibrantly generated symmetric monoidal model category. The category \mathcal{M}^I inherits the structure of a symmetric monoidal category over \mathcal{M} :

- (1) The internal tensor product $\otimes : \mathcal{M}^I \times \mathcal{M}^I \rightarrow \mathcal{M}^I$ is defined pointwise: $\forall i \in I, \forall X, Y \in \mathcal{M}^I, (X \otimes Y)(i) = X(i) \otimes Y(i)$.
- (2) We have a constant diagram functor

$$\begin{aligned} C : \mathcal{M} &\rightarrow \mathcal{M}^I \\ X &\mapsto C_X \end{aligned}$$

where $C_X(i) = X$, $C_X(i \rightarrow j) = id_X$, and for $f : X \rightarrow Y \in \mathcal{M}$, C_f is defined by $C_f(i) = f$. The external tensor product $\otimes : \mathcal{M} \times \mathcal{M}^I \rightarrow \mathcal{M}^I$ is given by $X \otimes F = C_X \otimes F$ for $X \in \mathcal{M}$, $F \in \mathcal{M}^I$.

- (3) The external hom $\text{Hom}_{\mathcal{M}^I}(-, -) : \mathcal{M}^I \times \mathcal{M}^I \rightarrow \mathcal{M}$ is given by

$$\text{Hom}_{\mathcal{M}^I}(X, Y) = \int_{i \in I} \text{Hom}_{\mathcal{M}}(X(i), Y(i))$$

where $\text{Hom}_{\mathcal{M}}(-, -)$ is the internal hom of \mathcal{M} .

We prove that \mathcal{M}^I with the model structure of theorem 3.3 satisfies the axiom MM0 and the axiom MM1 for the external tensor product. The axiom MM1 for

the internal tensor product of \mathcal{M}^I fails if we do not impose an extra assumption: we have to suppose that I admits finite coproducts.

Lemma 3.5. *The constant diagram functor $C : \mathcal{M} \rightarrow \mathcal{M}^I$ preserves (acyclic) cofibrations. Thus MM0 is satisfied in \mathcal{M}^I .*

Proof. Let $f : X \rightarrow Y$ be any (acyclic) cofibration in \mathcal{M} and $i \in I$. The functor $Mor_I(i, -) \otimes - : \mathcal{M} \rightarrow \mathcal{M}^I$ preserves generating (acyclic) cofibrations by definition of the cofibrantly generated model category structure on \mathcal{M}^I , so it preserves all (acyclic) cofibrations. Accordingly, the natural transformation $Mor_I(i, -) \otimes f$ is a (acyclic) cofibration of \mathcal{M}^I . For every $j \in I$ we have the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{inc} & \prod_{Mor_I(i,j)} X & \xrightarrow{pr_1} & X \\ C_f(j) \downarrow & & \downarrow Mor_I(i,j) \otimes f & & \downarrow C_f(j) \\ Y & \xrightarrow{inc} & \prod_{Mor_I(i,j)} Y & \xrightarrow{pr_1} & Y \end{array}$$

where $C_f(j) = f$ and $Mor_I(i, j) \otimes f = \prod_{Mor_I(i,j)} f$. The map inc is the inclusion and pr_1 the projection on the first component. This diagram gives rise to a commutative diagram in \mathcal{M}^I :

$$\begin{array}{ccccc} C_X & \xrightarrow{inc_X} & Mor_I(i, -) \otimes X & \xrightarrow{pr_1} & C_X \\ C_f \downarrow & & \downarrow Mor_I(i, -) \otimes f & & \downarrow C_f \\ C_Y & \xrightarrow{inc_Y} & Mor_I(i, -) \otimes Y & \xrightarrow{pr_1} & C_Y \end{array}$$

where $pr_1 \circ inc_X = id_{C_X}$ and $pr_1 \circ inc_Y = id_{C_Y}$. This means that C_f is a retract of $Mor_I(i, -) \otimes f$. A retract of an (acyclic) cofibration is an (acyclic) cofibration, so C_f is an (acyclic) cofibration. In particular, as MM0 is satisfied in \mathcal{M} , the unit $1_{\mathcal{M}}$ is cofibrant so $C_{1_{\mathcal{M}}}$ is cofibrant in \mathcal{M}^I . The object $C_{1_{\mathcal{M}}}$ is the unit of \mathcal{M}^I , which implies that the axiom MM0 holds in \mathcal{M}^I . \square

Lemma 3.6. *The axiom MM1 holds in \mathcal{M}^I for the external tensor product.*

Proof. Recall that C and C_a denote respectively the sets of generating cofibrations and generating acyclic cofibrations of \mathcal{M} . The sets F_C^I and $F_{C_a}^I$ denote the generating cofibrations and generating acyclic cofibrations of \mathcal{M}^I . According to corollary 4.2.5 in [13], it is sufficient to check the axiom MM1 for C and F_C^I , respectively C_a and $F_{C_a}^I$. We will just explain this verification for C and F_C^I , because the two other cases work in the same way. Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be two generating cofibrations of \mathcal{M} . Let $Mor_I(i, -) \otimes g$ be a generating cofibration of \mathcal{M}^I induced by g . We form their pushout-product for the external tensor product:

$$\begin{array}{ccc}
 A \otimes F_C^i & \xrightarrow{f \otimes id} & B \otimes F_C^i \\
 \downarrow id \otimes (Mor_I(i, -) \otimes g) & & \downarrow id \otimes (Mor_I(i, -) \otimes g) \\
 A \otimes F_D^i & \xrightarrow{\quad} & A \otimes F_D^i \amalg_{A \otimes F_C^i} B \otimes F_C^i \\
 & \searrow f \otimes id & \nearrow id \otimes (Mor_I(i, -) \otimes g) \\
 & & B \otimes F_D^i
 \end{array}$$

We take the evaluation of this diagram of diagrams in $j \in I$:

$$\begin{array}{ccc}
 A \otimes \amalg_{Mor_I(i, j)} C & \xrightarrow{\quad} & B \otimes \amalg_{Mor_I(i, j)} C \\
 \downarrow & & \downarrow \\
 A \otimes \amalg_{Mor_I(i, j)} D \rightrightarrows A \otimes \amalg_{Mor_I(i, j)} D \vee_{A \otimes \amalg_{Mor_I(i, j)} C} B \otimes \amalg_{Mor_I(i, j)} C & & \\
 \searrow & \nearrow & \\
 & & B \otimes \amalg_{Mor_I(i, j)} D
 \end{array}$$

The tensor product \otimes commutes with colimits so

$$A \otimes \amalg_{Mor_I(i, j)} D \vee_{A \otimes \amalg_{Mor_I(i, j)} C} B \otimes \amalg_{Mor_I(i, j)} C = \amalg_{Mor_I(i, j)} (A \otimes D \vee_{A \otimes C} B \otimes C)$$

and

$$B \otimes \amalg_{Mor_I(i, j)} D = \amalg_{Mor_I(i, j)} (B \otimes D).$$

Thus

$$(f_*, (Mor_I(i, -) \otimes g)_*)(j) = \amalg_{Mor_I(i, j)} (f_*, g_*),$$

for every $j \in J$, i.e

$$(f_*, (Mor_I(i, -) \otimes g)_*) = Mor_I(i, -) \otimes (f_*, g_*).$$

The morphism (f_*, g_*) is a cofibration because axiom MM1 holds in \mathcal{M} for the internal tensor product, and the functor $Mor_I(i, -) \otimes -$ preserves cofibrations, so $(f_*, (Mor_I(i, -) \otimes g)_*)$ is a cofibration in \mathcal{M}^I . \square

Remark 3.7. Lemma 3.6 implies axiom MM1', the “dual pushout-product” axiom, for the external hom $Hom_{\mathcal{M}^I}(-, -)$.

Lemma 3.8. *If I admits finite coproducts, then the axiom MM1 holds in \mathcal{M}^I for the internal tensor product.*

Proof. It is sufficient to check axiom MM1 in two cases: two morphisms of F_C^I , and two morphisms belonging respectively to F_C^I and $F_{C_a}^I$ (cf. [13], corollary 4.2.5). We just prove the first case, given that the second case can be treated in the same way. Let $Mor_I(i, -) \otimes f$ and $Mor_I(j, -) \otimes g$ be two generating cofibrations of \mathcal{M}^I obtained from the two generating cofibrations $f : A \rightarrowtail B$ and $g : C \rightarrowtail D$ of \mathcal{M} . We form their pushout-product for the internal tensor product:

$$\begin{array}{ccc}
 F_A^i \otimes F_C^j & \longrightarrow & F_B^i \otimes F_C^j \\
 \downarrow & & \downarrow \\
 F_A^i \otimes F_D^j & \longrightarrow & F_A^i \otimes F_D^j \amalg_{F_A^i \otimes F_C^i} F_B^i \otimes F_C^j \\
 & \searrow & \searrow \\
 & & F_B^i \otimes F_D^j
 \end{array}$$

We take the evaluation of this diagram of diagrams in $j \in I$:

$$\begin{array}{ccc}
 \amalg_{Mor_I(i,k)} A \otimes \amalg_{Mor_I(j,k)} C & \twoheadrightarrow & \amalg_{Mor_I(i,k)} B \otimes \amalg_{Mor_I(j,k)} C \\
 \downarrow & & \downarrow \\
 \amalg_{Mor_I(i,k)} A \otimes \amalg_{Mor_I(j,k)} D & \longrightarrow & pushout \\
 & \searrow & \searrow \\
 & & \amalg_{Mor_I(i,k)} B \otimes \amalg_{Mor_I(j,k)} D
 \end{array}$$

We have

$$\begin{aligned}
 pushout &= \amalg_{Mor_I(i,k) \times Mor_I(j,k)} (A \otimes D \bigvee_{A \otimes C} B \otimes C) \\
 &= \amalg_{Mor_I(i \vee j, k)} (A \otimes D \bigvee_{A \otimes C} B \otimes C)
 \end{aligned}$$

and

$$\amalg_{Mor_I(i,k)} B \otimes \amalg_{Mor_I(j,k)} D = \amalg_{Mor_I(i \vee j, k)} (B \otimes D)$$

so $((Mor_I(i, -) \otimes f)_*, (Mor_I(j, -) \otimes g)_*) = Mor_I(i \vee j, -) \otimes (f_*, g_*)$. The morphism (f_*, g_*) is a cofibration because the axiom MM1 holds in \mathcal{M} for the internal tensor product, and the functor $Mor_I(i \vee j, -) \otimes -$ preserves cofibrations, so $((Mor_I(i, -) \otimes f)_*, (Mor_I(j, -) \otimes g)_*)$ is a cofibration in \mathcal{M}^I . \square

From lemmas 3.5, 3.6 and 3.8 we conclude that:

Proposition 3.9. *Let \mathcal{M} be a cofibrantly generated symmetric monoidal model category and I a small category. If I admits finite coproducts, then \mathcal{M}^I forms a cofibrantly generated symmetric monoidal model category over \mathcal{M} .*

Before going to the heart of the matter, let us point out the fact that we can form the endomorphism prop of a diagram of diagrams. Transposing the construction proposed in the section 3.4.4 of [8] in the prop context, we get:

Definition 3.10. Let I, J be two small categories and $F : I \rightarrow \mathcal{M}^J$ a functor. We can define an *endomorphism prop* End_F in \mathcal{M} by

$$End_F(m, n) = \int_{i \in I} Hom_{\mathcal{M}^J}(F(i)^{\otimes m}, F(i)^{\otimes n})$$

where $\text{Hom}_{\mathcal{M}^J}(-, -)$ is the external hom of \mathcal{M}^J . The inner bifunctor of this formula is the endomorphism prop of the diagram $F(i)$ in \mathcal{M} .

Proposition 3.11. *Let P be a prop in \mathcal{M} . The props morphisms $P \rightarrow \text{End}_F$ are in bijection with the functorial P -actions on the objects $F(i) \in \mathcal{M}^J$, $i \in I$, such that $i \mapsto F(i)$ defines a functor $I \rightarrow (\mathcal{M}^J)^P$.*

3.2. Proof of proposition 3.1. We are now in position to begin the proof of proposition 3.1. Let \mathcal{E} be a symmetric monoidal cofibrantly generated model category over \mathcal{C} satisfying the limit monoid axioms. Let P be a cofibrant prop, defined in \mathcal{C} , with non-empty inputs (or outputs) and let $i : fw(\mathcal{E}^c)^P \hookrightarrow w(\mathcal{E}^c)^P$ be the inclusion of categories. The overall strategy is to show that for every $X \in (\mathcal{E}^{cf})^P$, the category $\mathcal{N}(X \downarrow i)$ is contractible and to apply Quillen's theorem A (cf. Quillen [21]). Let $fw(X \downarrow (\mathcal{E}^{cf})^P)$ denote the category whose objects are morphisms $X \rightarrow Y$ and morphisms are commutative triangles

$$\begin{array}{ccc} & & Y \\ & \nearrow & \downarrow \\ X & & \sim \\ & \searrow & \downarrow \\ & & Y' \end{array}$$

where $Y \xrightarrow{\sim} Y'$ is an acyclic fibration. By unraveling definitions, we see that $(X \downarrow i)$ is the full subcategory of $fw(X \downarrow (\mathcal{E}^{cf})^P)$ formed by the objects weakly equivalent to X . We will use the short notation $\mathcal{K} = (X \downarrow (\mathcal{E}^{cf})^P)$, and we set $\mathcal{L} = (X \downarrow \mathcal{E}^{cf})$ for the image of \mathcal{K} under the forgetful functor. We also consider $\mathcal{K}' = (X \downarrow i) = fw(X \downarrow w(\mathcal{E}^{cf})^P)$. The category \mathcal{K} admits an initial object $X \xrightarrow{\sim} X$, and \mathcal{K}' is a full subcategory of $fw\mathcal{K}$ including $X \xrightarrow{\sim} X$. The category of subdivisions of a simplicial set is the poset of its non degenerate simplices, where the partial order is given by the faces: a morphism in this category is a face map between two non degenerate simplices. Recall the following standard result about the simplicial nerve:

Lemma 3.12. *Let I, J be two small categories. Every simplicial map $\varphi : \mathcal{N}I \rightarrow \mathcal{N}J$ is induced by a functor $F : I \rightarrow J$, i.e the simplicial nerve functor $\mathcal{N} : \text{Cat} \rightarrow s\text{Set}$ is full (Cat is the category of small categories).*

Proof. The map φ defines F on the objects and morphisms, and we use the fact that φ commutes with faces and degeneracies to obtain the functoriality of F . \square

For the proof of proposition 3.1 we will need to apply proposition 3.9 to the category of diagrams \mathcal{L}^{sdK} , where sdK denotes the category of subdivisions of a simplicial set K . The idea is to use the model category structure of the props and the endomorphisms props of definition 3.10, in the case $\mathcal{M} = \mathcal{L}$ and $J = sdK$, to lift a P -algebra structure on a certain diagram in \mathcal{L}^{sdK} . We determine a coproduct of two non degenerate simplices for the poset structure of sdK in the case of a simplicial complex K . If two simplices α and β in sdK have no common face, then

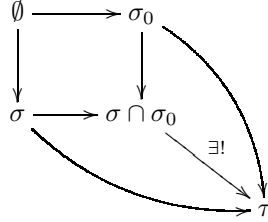
for every $\gamma \in sdK$, we have $Mor_{sdK}(\alpha, \gamma) \times Mor_{sdK}(\beta, \gamma) = \emptyset$ because at least one of these two sets of morphisms is empty, and in this case the pushout obtained in the proof of lemma 3.8 is empty. We have only to define a coproduct for two non degenerate simplices which have at least one common face. This is the subject of the following lemma:

Lemma 3.13. *We can define the coproduct of two non-degenerate simplices having at least one common face in the subdivision category of any simplicial complex.*

Proof. Recall that a simplicial complex is a particular case of simplicial set consisting of a finite collection of simplices K such that

- (i) if $\sigma \in K$ and τ is a face of σ then $\tau \in K$;
- (ii) for any $\sigma, \sigma_0 \in K$, the intersection $\sigma \cap \sigma_0$ is either the empty set or a common face of σ and σ_0 .

Let σ and σ_0 be two simplices of K having at least one common face. We then have $\sigma \cap \sigma_0 \neq \emptyset$, which implies that $\sigma \cap \sigma_0$ is a common face of σ and σ_0 (condition (ii)). We define their coproduct by $\sigma \vee \sigma_0 = \sigma \cap \sigma_0$. We easily check that this $\sigma \cap \sigma_0$ defines the coproduct of σ and σ_0 : $\sigma \cap \sigma_0$ is unique and satisfy the universal property of the coproduct



□

We want to prove that for every $n \in \mathbb{N}$, $\pi_n(|\mathcal{N}\mathcal{K}'|) = [S^n, |\mathcal{N}\mathcal{K}'|] = *$, where $|-|$ is the geometric realization functor and $[-, -]$ the homotopy classes. We consider, for every $n \in \mathbb{N}$, the simplicial complex $\partial\Delta^{n+1}$ as simplicial model of S^n . The simplicial complex $\partial\Delta^{n+1}$ is the boundary of Δ^{n+1} , i.e the simplicial complex obtained by withdrawing the $(n+1)$ -simplex of Δ^{n+1} . We prove in full generality the following result :

Proposition 3.14. *For any simplicial complex K_n , any simplicial map $\varphi : \mathcal{N}sdK_n \rightarrow \mathcal{N}\mathcal{K}'$ is null up to homotopy.*

Proof. We adapt the argument line of Rezk (lemma 4.2.5 in [22]). The main difference with his proof lies in the fact that we deal with the absence of model category structure on the category of P -algebras (recall that the lemma 4.2.5 in [22] involves model category devices). Therefore we use proposition 3.9, proposition 3.11 and lemma 3.13 to apply lifting techniques from [7] and get the desired P -algebras structures on our diagrams. According to lemma 3.12, it is sufficient to prove that any functor $F : sdK_n \rightarrow \mathcal{K}'$ induces a map between the simplicial nerves null up to homotopy. Let F be such a functor. In order to use the factorization axioms of a model category, we work temporarily in \mathcal{L}^{sdK_n} and note also F the composite

functor $U \circ F$ taking values in $U(\mathcal{K}') \subset \mathcal{L}$ and thus belonging to \mathcal{L}^{sdK_n} . The initial diagram \bar{X} is the constant diagram in $X \rightrightarrows X$. To simplify, we will abbreviate the notation of a morphism $X \rightarrow Y$, regarded as an object of \mathcal{L} , to the single target object Y , assuming that any such given Y comes together with a canonical morphism $X \rightarrow Y$. We similarly abbreviate the notation of a morphism between $X \rightarrow Y$ and $X \rightarrow Y'$, which is a commutative triangle, to the morphism $Y \rightarrow Y'$ between the targets. The functor $F \times \bar{X}$ is defined on the objects by $(F \times \bar{X})(k) = F(k) \times X$ and on the morphisms by $(F \times \bar{X})(k_1 \xrightarrow{\phi} k_2) = F(k_1) \times X \xrightarrow{F(\phi) \times id_X} F(k_2) \times X$. For every $\phi : k_1 \rightarrow k_2 \in sdK_n$, $F(\phi)$ is an acyclic fibration so $F(\phi) \times id_X$ too, thus $F \times \bar{X}$ takes actually its values in $fw\mathcal{L}$. Recall that \mathcal{L} inherits a cofibrantly generated model category structure, according to Hirschhorn [12]. Therefore \mathcal{L}^{sdK_n} is endowed with a model category structure, so that we can give a decomposition of the unique map $\bar{X} \rightarrow F \times \bar{X}$ in an acyclic cofibration followed by a fibration:

$$\bar{X} \xrightarrow{\sim} G \rightarrow F \times \bar{X}.$$

It gives us a diagram \mathcal{Y}

$$\begin{array}{ccc}
 & & \bar{X} \\
 & \nearrow^{\quad \quad \quad} & \\
 \bar{X} & \xrightarrow[\quad i \quad]{\quad \sim \quad} & G \\
 & \searrow_{\quad \quad \quad} & \\
 & & F
 \end{array}$$

The map $(p_1, p_2) : G \rightarrow F \times \bar{X}$ is a fibration so for every $k \in sdK_n$, $(p_1, p_2)(k)$ is a fibration. Moreover, $F(k)$ and X are fibrant so $p_1(k)$ and $p_2(k)$ are acyclic fibrations (cf [4], lemma 4.4). By definition of the weak equivalences and fibrations in a diagram category, we conclude that p_1 and p_2 are acyclic fibrations in $U(\mathcal{M})^{sdK_n}$. Now, considering again F and \bar{X} as functors $sdK_n \rightarrow \mathcal{K}' \subset \mathcal{K}$, we want to put a P -algebra structure on \mathcal{Y} which preserves the P -algebra structures existing on F and \bar{X} . To be more explicit, we want to produce a lifting

$$\begin{array}{ccc}
 0 & \xrightarrow{\quad} & End_{\mathcal{Y}} \\
 \downarrow & \nearrow \quad \quad \quad & \downarrow \sim \\
 P & \xrightarrow{\quad} & End_{\mathcal{V}}
 \end{array}$$

where \mathcal{V} is the subdiagram of \mathcal{Y} consisting of the two external arrows obtained by withdrawing G , i , p_1 and p_2 from \mathcal{Y} . The idea is to show that we have an acyclic fibration $End_{\mathcal{Y}} \xrightarrow{\sim} End_{\mathcal{V}}$ and to use the cofibrancy of P to produce the lifting. The category \mathcal{L} forms a cofibrantly generated symmetric monoidal model category over \mathcal{C} satisfying the limit monoid axioms, as does \mathcal{E} . We examine the case of \mathcal{L}^{sdK_n} . We know that it possesses a cofibrantly generated model category structure. It is a symmetric monoidal category over \mathcal{E} , which is a symmetric monoidal category over \mathcal{C} , so \mathcal{L}^{sdK_n} is also a symmetric monoidal category over \mathcal{C} . Moreover, \mathcal{L}^{sdK_n} satisfies the properties described in proposition 1.8, given that these properties are satisfied in \mathcal{L} and that the fibrations of \mathcal{L}^{sdK_n} are defined pointwise. Lemmas 3.5

and 3.6 ensure that the axioms MM0 and MM1 for the external tensor product hold in \mathcal{L}^{sdK_n} . The fact that K_n is a simplicial complex and lemma 3.13 allows us to apply lemma 3.8, i.e to check the axiom MM1 for the internal tensor product in \mathcal{L}^{sdK_n} . We have now verified all the necessary conditions to apply the proof of lemma 8.3 in [7] to obtain the desired acyclic fibration.

Finally, we want the diagram of functors $F \xleftarrow{\sim} G \xrightarrow{\sim} \bar{X}$ to live in \mathcal{K}^{sdK_n} . We have already checked that F and \bar{X} take their values in \mathcal{K}' . The maps p_1 and p_2 are natural transformations in \mathcal{K}^{sdK_n} . So it remains to prove that G takes its values in \mathcal{K}' . For this, we refer the reader to the proof of lemma 4.2.5 in [22]. This diagram of functors induces simplicial homotopies $\mathcal{N}F \sim \mathcal{N}G \sim 0$, that is what we expected. \square

Recall that $|\mathcal{N}sdK_n| \simeq |K_n|$. It includes especially the case $K_n = \partial\Delta^{n+1}$. We deduce that the nerve $\mathcal{N}\mathcal{K}' = \mathcal{N}(X \downarrow i)$ is contractible for every $X \in (\mathcal{E}^{cf})^P$, so $i : fw(\mathcal{E}^c)^P \hookrightarrow w(\mathcal{E}^c)^P$ gives rise to a weak equivalence of nerves according to Quillen's theorem A. It concludes the proof of proposition 3.1.

4. EXTENSION OF THE RESULTS IN THE COLORED PROP SETTING

Definition 4.1. Let C be a non-empty set, called the *set of colors*, and \mathcal{C} a symmetric monoidal category.

(1) A *C-colored Σ -biobject* M is a double sequence of objects $\{M(m, n) \in \mathcal{E}\}_{(m, n) \in \mathbb{N}^2}$ where each $M(m, n)$ admits commuting left Σ_m action and right Σ_n action as well as a decomposition

$$M(m, n) = \text{colim}_{c_i, d_i \in C} M(c_1, \dots, c_m; d_1, \dots, d_n)$$

compatible with these actions. The objects $M(c_1, \dots, c_m; d_1, \dots, d_n)$ should be thought as spaces of operations with colors c_1, \dots, c_m indexing the m inputs and colors d_1, \dots, d_n indexing the n outputs.

(2) A *C-colored prop* P is a *C-colored Σ -biobject* endowed with a horizontal composition

$$\begin{aligned} \circ_h : P(c_{11}, \dots, c_{1m_1}; d_{11}, \dots, d_{1n_1}) \otimes \dots \otimes P(c_{k1}, \dots, c_{km_k}; d_{k1}, \dots, d_{kn_k}) \rightarrow \\ P(c_{11}, \dots, c_{km_k}; d_{k1}, \dots, d_{kn_k}) \subseteq P(m_1 + \dots + m_k, n_1 + \dots + n_k) \end{aligned}$$

and a vertical composition

$$\circ_v : P(c_1, \dots, c_k; d_1, \dots, d_n) \otimes P(a_1, \dots, a_m; b_1, \dots, b_k) \rightarrow P(a_1, \dots, a_m; d_1, \dots, d_n) \subseteq P(m, n)$$

which is equal to zero unless $b_i = c_i$ for $1 \leq i \leq k$. These two compositions satisfy associativity axioms (we refer the reader to [14] for details).

Definition 4.2. (1) Let $\{X_c\}_C$ be a collection of objects of \mathcal{E} . The *C-colored endomorphism prop* $\text{End}_{\{X_c\}_C}$ is defined by

$$\text{End}_{\{X_c\}_C}(c_1, \dots, c_m; d_1, \dots, d_n) = \text{Hom}_{\mathcal{E}}(X_{c_1} \otimes \dots \otimes X_{c_m}, X_{d_1} \otimes \dots \otimes X_{d_n})$$

with a horizontal composition given by the tensor product of homomorphisms and a vertical composition given by the composition of homomorphisms with matching colors.

(2) Let P be a C -colored prop. A P -algebra is the data of a collection of objects $\{X_c\}_C$ and a C -colored prop morphism $P \rightarrow \text{End}_{\{X_c\}_C}$.

Example 4.3. Let I be a small category, P a prop in \mathcal{C} . We can build an $ob(I)$ -colored prop P_I such that the P_I -algebras are the I -diagrams of P -algebras in \mathcal{E} in the same way as that of [17].

To endow the category of colored props with a model category structure, the cofibrantly generated symmetric monoidal model structure on \mathcal{C} is not sufficient. We have to suppose moreover that the domains of the generating cofibrations and acyclic generating cofibrations are small (cf [11], 10.4.1), that is to say the model structure is strongly cofibrantly generated:

Theorem 4.4. (cf. [14], theorem 1.1) *Let C be a non-empty set. Let \mathcal{C} be a strongly cofibrantly generated symmetric monoidal model category with a symmetric monoidal fibrant replacement functor, and either:*

- (1) *a cofibrant unit and a cocommutative interval, or*
- (2) *functorial path data.*

Then the category \mathcal{P}_C of C -colored props in \mathcal{C} forms a strongly cofibrantly generated model category with fibrations and weak equivalences defined componentwise in \mathcal{C} .

This theorem works especially with the categories of simplicial sets, simplicial modules over a commutative ring and chain complexes over a characteristic 0 ring (our main category in this paper).

This model structure is similar to that of 1-colored props, and we can define C -colored endomorphism props of morphisms (see [14], section 4) and more generally of any kind of diagram, so the lifting properties used in the two previous sections works in the C -colored case. Moreover, in the proof of theorem 0.1, the replacement of the operations $X^{\otimes m} \rightarrow X^{\otimes n}$ by $P(m, n)$ can be done using a C -colored prop P instead of a 1-colored one without changing anything to the proof, therefore we finally get the C -colored version of theorem 0.1 and proposition 3.1. We do not have to change anything to theorem 0.1, given that $Ch_{\mathbb{K}}$ satisfies the hypotheses of theorem 4.4. Concerning proposition 3.1, we just have to suppose that \mathcal{C} verifies the additional hypotheses of theorem 4.4.

5. APPLICATION: THE MODULI SPACE OF P -ALGEBRA STRUCTURES AS A HOMOTOPY FIBER

5.1. Moduli spaces of algebra structures over a prop. A moduli space of algebra structures over a prop P , on a given object X of \mathcal{E} , is a simplicial set whose points are the prop morphisms $P \rightarrow \text{End}_X$. Such a moduli space can be more generally defined on diagrams of \mathcal{E} . We then deal with endomorphism props of diagrams. To construct properly such a simplicial set and give its first fundamental properties, we have to recall some results about the cosimplicial resolutions in a model category. For the sake of brevity and clarity, we refer the reader to the chapter 16 in [11] for a complete treatment of the notions of simplicial resolutions, cosimplicial resolutions and Reedy model categories.

Definition 5.1. Let \mathcal{M} be a model category and let X be an object of \mathcal{M} .

(1) A *cosimplicial resolution* of X is a cofibrant approximation to the constant cosimplicial object cc_*X in the Reedy model category structure on cosimplicial objects \mathcal{M}^Δ of \mathcal{M} .

(2) A *simplicial resolution* of X is a fibrant approximation to the constant simplicial object cs_*X in the Reedy model category structure on simplicial objects $\mathcal{M}^{\Delta^{op}}$ of \mathcal{M} .

Definition 5.2. Let \mathcal{M} be a model category and let X be an object of \mathcal{M} .

(1) A *cosimplicial frame* on X is a cosimplicial object \tilde{X} in \mathcal{X} , together with a weak equivalence $\tilde{X} \rightarrow cc_*X$ in the Reedy model category structure of \mathcal{M}^Δ . It has to satisfy the two following properties : the induced map $\tilde{X}^0 \rightarrow X$ is an isomorphism, and if X is cofibrant in \mathcal{M} then \tilde{X} is cofibrant in \mathcal{M}^Δ .

(2) A *simplicial frame* on X is a simplicial object \tilde{X} in \mathcal{X} , together with a weak equivalence $cs_*X \rightarrow \tilde{X}$ in the Reedy model category structure of \mathcal{M}^Δ . It has to satisfy the two following properties : the induced map $X \rightarrow \tilde{X}^0$ is an isomorphism, and if X is fibrant in \mathcal{M} then \tilde{X} is fibrant in $\mathcal{M}^{\Delta^{op}}$.

Proposition 5.3. (cf [11], proposition 16.1.9) *Let \mathcal{M} be a model category. There exists functorial simplicial resolutions and functorial cosimplicial resolutions in \mathcal{M} .*

Proposition 5.4. (cf. [11], corollaries 16.5.3 and 16.5.4) *Let \mathcal{M} be a model category and C a cosimplicial resolution in \mathcal{M} .*

- (1) *If X is a fibrant object of \mathcal{M} , then $Mor_{\mathcal{M}}(C, X)$ is a fibrant simplicial set.*
- (2) *If $p : X \rightarrow Y$ is a fibration in \mathcal{M} , then $p_* : Mor_{\mathcal{M}}(C, X) \rightarrow Mor_{\mathcal{M}}(C, Y)$ is a fibration of simplicial sets, acyclic if p is so.*
- (3) *If $p : X \xrightarrow{\sim} Y$ is a weak equivalence of fibrant objects in \mathcal{M} , then $p_* : Mor_{\mathcal{M}}(C, X) \rightarrow Mor_{\mathcal{M}}(C, Y)$ is a weak equivalence of fibrant simplicial sets.*

Proposition 5.5. (cf. [11], proposition 16.6.3) *Let X be an object of \mathcal{M} .*

- (1) *If X is cofibrant then every cosimplicial frame of X is a cosimplicial resolution of X .*
- (2) *If X is fibrant then every simplicial frame of X is a simplicial resolution of X .*

Now let P be a cofibrant prop with non-empty inputs in a cofibrantly generated symmetric monoidal model category \mathcal{C} . The cosimplicial frame $P \otimes \Delta[-]$ on P given by $(P \otimes \Delta[-])(m, n) = P(m, n) \otimes \Delta[-]$ is a cosimplicial resolution of P according to proposition 5.5.

Definition 5.6. Let \mathcal{E} be a symmetric monoidal model category over \mathcal{C} and P a cofibrant prop with non-empty inputs in \mathcal{C} . Let I be a small category and $\{X_i\}_{i \in I}$ a I -diagram in \mathcal{E} . The *moduli space of P -algebra structures* on $\{X_i\}_{i \in I}$ is the simplicial set defined by

$$P\{X_i\}_{i \in I} = Mor_{\mathcal{P}_0}(P \otimes \Delta[-], End_{\{X_i\}_{i \in I}}).$$

We get a functor

$$\begin{aligned} \mathcal{P}_0 &\rightarrow sSet \\ P &\mapsto P\{X_i\}_{i \in I}. \end{aligned}$$

We can already get two interesting properties of these moduli spaces:

Lemma 5.7. *Suppose moreover that \mathcal{E} satisfies the limit monoid axioms. If X is a fibrant and cofibrant object of \mathcal{E} , then $P\{X\}$ is a fibrant simplicial set.*

Proof. If X is fibrant and cofibrant, we can show by arguments based on the pushout-product axiom and the limit monoid axiom LM2 that End_X is a fibrant prop (cf lemma 7.2 in [7]), so $P\{X\}$ is fibrant according to proposition 5.4. \square

Lemma 5.8. *Suppose that \mathcal{E} satisfies the limit monoid axioms. Let X be a fibrant and cofibrant object of \mathcal{E} . Every weak equivalence of cofibrant props $P \xrightarrow{\sim} Q$ gives rise to a weak equivalence of fibrant simplicial sets (which is a homotopy equivalence, since every object is automatically cofibrant in the model category structure of simplicial sets) $Q\{X\} \xrightarrow{\sim} P\{X\}$.*

Proof. Let $\varphi : P \rightarrow Q$ be a weak equivalence of cofibrant props. According to proposition 16.1.24 of [11], the map φ induces a Reedy weak equivalence of cosimplicial resolutions $P \otimes \Delta[-] \xrightarrow{\sim} Q \otimes \Delta[-]$. The object X is fibrant and cofibrant so End_X is fibrant, and we conclude by corollary 16.5.5 of [11] that $P \otimes \Delta[-] \xrightarrow{\sim} Q \otimes \Delta[-]$ induces $Q\{X\} \xrightarrow{\sim} P\{X\}$. \square

5.2. Moduli spaces of algebra structures on fibrations. We start by recalling lemma 7.2 of [7]. Let $f : A \rightarrow B$ be a morphism of \mathcal{E} , we have a pullback

$$\begin{array}{ccc} End_{\{A \rightarrow f B\}} & \xrightarrow{d_0} & End_B \\ d_1 \downarrow & & \downarrow f^* \\ End_A & \xrightarrow{f_*} & Hom_{AB} \end{array}$$

where Hom_{AB} is defined by $Hom_{AB}(m, n) = Hom_{\mathcal{E}}(A^{\otimes m}, B^{\otimes n})$.

Lemma 5.9. (cf. [7], lemma 7.2) *Suppose that A and B are fibrant and cofibrant. Then End_A and End_B are fibrant props. Moreover:*

- (1) *If f is a (acyclic) fibration then so is d_0 .*
- (2) *If f is a cofibration, then d_1 is a fibration. If f is also acyclic then d_1 is an acyclic fibration and d_0 a weak equivalence.*

Remark 5.10. It is a generalization in the prop context of propositions 4.1.7 and 4.1.8 of [22].

Lemma 5.11. *Let $X_n \twoheadrightarrow \dots \twoheadrightarrow X_1 \twoheadrightarrow X_0$ be a chain of fibrations in \mathcal{E}^{cf} (the full subcategory of \mathcal{E} consisting of objects which are both fibrant and cofibrant). For every $0 \leq k \leq n-1$, the map d_0 in the pullback*

$$\begin{array}{ccc} \text{End}_{\{X_n \twoheadrightarrow \dots \twoheadrightarrow X_0\}} & \xrightarrow{d_0} & \text{End}_{\{X_k \twoheadrightarrow \dots \twoheadrightarrow X_0\}} \\ d_1 \downarrow & & \downarrow \\ \text{End}_{\{X_n \twoheadrightarrow \dots \twoheadrightarrow X_{k+1}\}} & \longrightarrow & \text{Hom}_{X_{k+1}X_k} \end{array}$$

is a fibration. Moreover, if the fibrations in the chain $X_n \twoheadrightarrow \dots \twoheadrightarrow X_1 \twoheadrightarrow X_0$ are acyclic then so is d_0 .

Proof. We prove this lemma by induction. For $n = 1$ it is lemma 5.9. Now suppose that our lemma is true for a given integer $n \geq 1$. Let $X_{n+1} \twoheadrightarrow \dots \twoheadrightarrow X_1 \twoheadrightarrow X_0$ be a chain of fibrations in \mathcal{E}^{cf} . We distinguish two cases:

-the case $k = n$: we have the pullback

$$\begin{array}{ccc} \text{End}_{\{X_{n+1} \twoheadrightarrow \dots \twoheadrightarrow X_0\}} & \xrightarrow{d_0} & \text{End}_{\{X_n \twoheadrightarrow \dots \twoheadrightarrow X_0\}} \\ d_1 \downarrow & & \downarrow \\ \text{End}_{X_{n+1}} & \xrightarrow{f_*} & \text{Hom}_{X_{n+1}X_n} \end{array}$$

where $f : X_{n+1} \twoheadrightarrow X_n$. The fact that f is a fibration implies that f_* is a fibration, so d_0 is a fibration because of the stability of fibrations under pullback, and the acyclicity of f implies the acyclicity of d_0 . The detailed proof of these statements is done in the proof of lemma 7.2 in [7].

-the case $0 \leq k \leq n-1$: $d_0 = \text{End}_{\{X_{n+1} \twoheadrightarrow \dots \twoheadrightarrow X_0\}} \rightarrow \text{End}_{\{X_n \twoheadrightarrow \dots \twoheadrightarrow X_0\}} \rightarrow \text{End}_{\{X_k \twoheadrightarrow \dots \twoheadrightarrow X_0\}}$ is the composite of a map satisfying the induction hypothesis with the map of the case $k = n$, so the conclusion follows. \square

Remark 5.12. This lemma is the generalization in the prop context of proposition 4.1.9 of [22].

We deduce from lemmas 5.9 and 5.11 the following properties of our moduli spaces:

Proposition 5.13. *Let $f : X \rightarrow Y$ be a morphism of \mathcal{E}^{cf} and P a cofibrant prop in \mathcal{C} . The pullback of lemma 5.9 gives rise to the following diagram of simplicial sets:*

$$P\{X\} \xleftarrow{(d_1)_*} P\{f\} \xrightarrow{(d_0)_*} P\{Y\}$$

(1) *If f is a cofibration then $(d_1)_*$ is a fibration. Moreover, if f is acyclic then $(d_0)_*$ and $(d_1)_*$ are weak equivalences.*

(2) *If f is a fibration then $(d_0)_*$ is a fibration. Moreover, if f is acyclic then $(d_0)_*$ and $(d_1)_*$ are weak equivalences.*

Proof. (1) If f is a cofibration then d_1 is a fibration. So $(d_1)_*$ is a fibration of simplicial sets according to proposition 5.4. If f is acyclic, then d_0 and d_1 are weak equivalences. The objects X and Y are fibrant and cofibrant so End_X and End_Y are fibrant props, which implies that $End_{\{f\}}$ is also fibrant. We deduce from this and proposition 5.4 that $(d_0)_*$ and $(d_1)_*$ are weak equivalences.

(2) The proof is the same as in the previous case. \square

By induction we can also prove the following proposition:

Proposition 5.14. *Let $X_n \xrightarrow{\sim} \dots \xrightarrow{\sim} X_1 \xrightarrow{\sim} X_0$ be a chain of acyclic fibrations in \mathcal{E}^{cf} and P a cofibrant prop in \mathcal{C} . For every $0 \leq k \leq n-1$, the map $(d_0)_*$ is an acyclic fibration and $(d_1)_*$ a weak equivalence in the diagram below:*

$$P\{X_n \xrightarrow{\sim} \dots \xrightarrow{\sim} X_{k+1}\} \xleftarrow{(d_1)_*} P\{X_n \xrightarrow{\sim} \dots \xrightarrow{\sim} X_0\} \xrightarrow{(d_0)_*} P\{X_k \xrightarrow{\sim} \dots \xrightarrow{\sim} X_1\}.$$

Remark 5.15. Propositions 5.13 and 5.14 are generalizations in the prop context of propositions 4.1.11, 4.1.12 and 4.1.13 in [22].

5.3. Proof of theorem 0.2. We have now all the key results to generalize Rezk's theorem to algebras over props and colored props. The remaining arguments are the same as that of Rezk, so we will not repeat it with all details but essentially show how our theorem 0.1 and proposition 3.1 fit in the proof.

Let \mathcal{P} a cofibrant prop, and $\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[-] \otimes P}$ the bisimplicial set defined by $(\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[-] \otimes P})_{m,n} = (\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[n] \otimes P})_m$. The prop P is cofibrant, thus so is $\Delta[n] \otimes P$ for every $n \geq 0$. According to proposition 3.1, we have a weak équivalence induced by an inclusion of categories

$$\mathcal{N}fw(\mathcal{E}^{cf})^{\Delta[n] \otimes P} \xrightarrow{\sim} \mathcal{N}w(\mathcal{E}^{cf})^{\Delta[n] \otimes P}$$

Moreover, for every $n, n' \geq 0$, $\Delta[n] \rightarrow \Delta[n']$ induces a weak equivalence of cofibrant props $\Delta[n] \otimes P \rightarrow \Delta[n'] \otimes P$ and thereby a weak equivalence of simplicial sets

$$\mathcal{N}w(\mathcal{E}^{cf})^{\Delta[n'] \otimes P} \xrightarrow{\sim} \mathcal{N}w(\mathcal{E}^{cf})^{\Delta[n] \otimes P}$$

according to theorem 0.1. We obtain a zigzag of weak equivalences

$$diag \mathcal{N}fw(\mathcal{E}^{cf})^{\Delta[-] \otimes P} \xrightarrow{\sim} diag \mathcal{N}w(\mathcal{E}^{cf})^{\Delta[-] \otimes P} \xleftarrow{\sim} \mathcal{N}w(\mathcal{E}^{cf})^P$$

We use an adaptation of a slightly modified version of Quillen's theorem B (cf. [21]), namely lemma 4.2.2 in [22], in order to determine the homotopy fiber of the map $diag \mathcal{N}fw(\mathcal{E}^{cf})^{\Delta[-] \otimes P} \rightarrow \mathcal{N}fw(\mathcal{E}^{cf})$. To prove that our map verifies the hypotheses of this lemma we use the propositions of section 5.2 exactly in the same way as Rezk in the operadic case. Then we check that $diag(U \downarrow X) \simeq P\{X\}$ where $U : fw(\mathcal{E}^{cf})^{\Delta[-] \otimes P} \rightarrow fw\mathcal{E}^{cf}$ is the forgetful functor (by using again the propositions of section 5.2) and finally we get the following diagram:

$$\begin{array}{ccccc}
 P\{X\} & \longrightarrow & \text{diag} \mathcal{N}fw(\mathcal{E}^{cf})^{\Delta[-] \otimes P} & \xrightarrow{\sim} & \text{diag} \mathcal{N}w(\mathcal{E}^{cf})^{\Delta[-] \otimes P} \xleftarrow{\sim} \mathcal{N}w(\mathcal{E}^{cf})^P . \\
 \downarrow & & \downarrow & & \downarrow \\
 pt & \longrightarrow & \mathcal{N}(fw\mathcal{E}^{cf}) & \xrightarrow{\sim} & \mathcal{N}(w\mathcal{E}^{cf})
 \end{array}$$

The proof of theorem 0.2 is complete.

Given the model category structure on the colored props in $Ch_{\mathbb{K}}$, the construction of moduli spaces from cosimplicial framings makes sense. We can obtain the colored prop version of theorem 0.2 by replacing the cofibrant prop P by a colored cofibrant prop (According to the section 4 of our paper, theorem 0.1 and proposition 3.1 have their equivalent in the colored prop setting, and so do the propositions of section 5.2 according to [14], section 4):

Theorem 5.16. *(generalization of [22], theorem 1.1.5, in the case of colored props)* Let \mathcal{E} be a cofibrantly generated symmetric monoidal model category over $Ch_{\mathbb{K}}$ satisfying the limit monoid axioms. Let P be a cofibrant C -colored prop in $Ch_{\mathbb{K}}$, where C is a non-empty set, and $\{X_c\}_{c \in C}$ a collection of objects of \mathcal{E}^{cf} . Then the commutative square

$$\begin{array}{ccc}
 P\{X_c\}_{c \in C} & \longrightarrow & \mathcal{N}(w(\mathcal{E}^{cf})^P) \\
 \downarrow & & \downarrow \\
 \{X_c\}_{c \in C} & \longrightarrow & \mathcal{N}(w\mathcal{E}^{cf})
 \end{array}$$

is a homotopy pullback of simplicial sets.

Remark 5.17. Note that we can recover the transfer theorem of bialgebras structures obtained in [7] (theorem A) and its colored version as a consequence of theorem 5.16. Indeed, let P be a cofibrant prop in $Ch_{\mathbb{K}}$. Let $X \xrightarrow{\sim} Y$ be a morphism of \mathcal{E}^{cf} such that Y is endowed with a P -algebra structure. We have a homotopy pullback of simplicial sets

$$\begin{array}{ccc}
 P\{X\} & \xrightarrow{p} & \mathcal{N}(w(\mathcal{E}^{cf})^P) \\
 \downarrow & & \downarrow \mathcal{N}U \\
 \{X\} & \longrightarrow & \mathcal{N}(w\mathcal{E}^{cf})
 \end{array}$$

which induces an exact sequence of pointed sets

$$\pi_0 P\{X\} \rightarrow \pi_0 \mathcal{N}(w(\mathcal{E}^{cf})^P) \rightarrow \pi_0 \mathcal{N}(w\mathcal{E}^{cf}).$$

The base point of the set $\pi_0 \mathcal{N}(w\mathcal{E}^{cf})$ is the weak equivalence class of X , denoted by $[X]$. The weak equivalence $X \xrightarrow{\sim} Y$ in \mathcal{E}^{cf} implies that we have the equality $[Y] = [X]$ and thus $\pi_0 \mathcal{N}U([Y]_P) = [X]$, where $[Y]_P$ is the weak equivalence class of Y in $(\mathcal{E}^{cf})^P$. The exactness of the sequence above implies that $\pi_0 p(P\{X\}) = (\pi_0 \mathcal{N}U)^{-1}([X])$ so $[Y]_P \in \pi_0 p(P\{X\})$. It means that there exists a P -algebra structure on X such that we have a zigzag of P -algebras morphisms

$$X \xleftarrow{\sim} \dots \xrightarrow{\sim} Y$$

which are weak equivalences of \mathcal{E}^{cf} .

Remark 5.18. Theorem 5.16 applies especially in the following case. Let I be a small category, P a prop in $Ch_{\mathbb{K}}$, and P_I the $ob(I)$ -colored prop such that the P_I -algebras are the I -diagrams of P -algebras in $Ch_{\mathbb{K}}$ (see example 37). Let us take a cofibrant replacement of P_I , namely $(P_I)_{\infty}$, then we have the following homotopy pullback of simplicial sets for any collection $\{X_i\}$ of chain complexes indexed by $ob(I)$:

$$\begin{array}{ccc} (P_I)_{\infty}\{X_i\} & \longrightarrow & \mathcal{N}(w(Ch_{\mathbb{K}})^P) \\ \downarrow & & \downarrow \\ pt & \longrightarrow & \mathcal{N}(wCh_{\mathbb{K}}) \end{array}$$

(recall that every chain complex over a field of characteristic zero is fibrant and cofibrant, so there is no need of writing this hypothesis here). Moreover, given that a weak equivalence between two cofibrant props induces a weak equivalence between the associated classifying spaces, we can take any cofibrant replacement of P_I to get the same classifying space up to weak equivalence.

REFERENCES

- [1] J. Boardman, R. Vogt, *Homotopy-everything H-spaces*, Bull. Amer. Math. Soc. 74 (1968), 1117-1122.
- [2] J. Boardman, R. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics 347, Springer-Verlag, 1973.
- [3] I. Ciocan-Fontanine, M. M. Kapranov, *Derived Hilbert schemes*, J. Amer. Math. Soc. 15 (2002), 787-815.
- [4] W. G. Dwyer, J. Spalinski, *Homotopy theories and model categories*, Handbook of Algebraic Topology (1995), 73-126.
- [5] W. G. Dwyer, D. M. Kan, *Function complexes in homotopical algebra*, Topology 19 (1980), pp. 427-440.
- [6] Benjamin Enriquez, Pavel Etingof, *On the invertibility of quantization functors*, Journal of Algebra volume 289 (2005), 321-345.
- [7] Benoit Fresse, *Props in model categories and homotopy invariance of structures*, Georgian Math. J. 17 (2010), pp. 79-160.
- [8] Benoit Fresse, *Modules over operads and functors*, Lecture Notes in Mathematics 1967, Springer-Verlag (2009).
- [9] Benoit Fresse, *The bar complex of an E-infinity algebra*, Adv. Math. 223 (2010), pp. 2049-2096.
- [10] Paul G. Goerss, John F. Jardine, *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Verlag (1999).
- [11] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs volume 99, AMS (2003).
- [12] Philip S. Hirschhorn, *Overcategories and undercategories of model categories*, a complement of Model categories and their localizations available at <http://www-math.mit.edu/~psh/#Undercat>.
- [13] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs volume 63, AMS (1999).
- [14] Mark W. Johnson, Donald Yau, *On homotopy invariance for algebras over colored PROPs*, Journal of Homotopy and Related Structures 4 (2009), 275-315.
- [15] Saunders MacLane, *Categories for the working mathematician*, Graduate texts in Mathematics, second edition, Springer Verlag (1998).
- [16] Saunders MacLane, *Categorical algebra*, Bull. Amer. Math. Soc. Volume 71, Number 1 (1965), 40-106.
- [17] Martin Markl, *Homotopy algebras are homotopy algebras*, Forum Math. 16 (2004), 129-160.

- [18] Martin Markl, *Intrinsic brackets and the L_∞ -deformation theory of bialgebras*, Journal of Homotopy and related structures (2010).
- [19] P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics 271, Springer-Verlag, 1972.
- [20] Serguei Merkulov, Bruno Vallette, *Deformation theory of representations of properads II*, Journal für die reine und angewandte Mathematik (Crelles Journal), Issue 636 (2009).
- [21] Daniel G. Quillen, *Higher algebraic K-theory I*, in Higher K-theories, Vol. I, Lecture Notes in Mathematics 341, Springer-Verlag (1973), pp. 85-147.
- [22] Charles Rezk, *Spaces of algebra structures and cohomology of operads*, PhD Thesis, Massachusetts Institute of Technology (1996).
- [23] Michael Schlessinger, James Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, Journal of Pure and Applied Algebra 38 (1985), 313-322.

LABORATOIRE PAUL PAINLEVÉ, UNIVERSITÉ DE LILLE 1, CITÉ SCIENTIFIQUE, 59655 VILLENEUVE D'ASCQ CEDEX, FRANCE
E-mail address: `Sinan.Yalin@ed.univ-lille1.fr`